ELASTIC FIELD IN COMPOSITE ROTATING BODIES

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BY

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To my mother

CERTIFICATE

Certified that the work contained in this thesis has been carried out under my supervision and that the work has not been submitted elsewhere for a degree.

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Vsrao

(V. S. Rao)

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LIST OF FREQUENTLY OCCURING SYMBOLS

x,y,z	Rectangular Cartesian coordinates
r,θ	Plane Polar coordinates
r,θ,ϕ	Spherical Polar coordinates
ξ, η	Elliptic coordinates
u,v,w	Displacements in x,y,z directions
P_{xx},P_{yy},P_{zz}	Normal stresses in Cartesian coordinates
P_{xy},P_{yz},P_{zx}	Shearing stresses in Cartesian coordinates
e_{xx},e_{yy},e_{zz}	Extensional strains
e_{xy},e_{yz},e_{zx}	Shearing strains
E	Young's Modulus
ν	Poisson's ratio
λ,μ	Lam'e constants
F_x,F_y,F_z	Components of body force in x,y,z directions
$\phi_1(z),\psi_1(z)$	Complex potential functions in the insert
$\phi_s(z),\psi_s(z)$	Complex potential functions in the shell
t,σ	Boundary points
∇^2	Laplacian operator in two dimensions
k	$(3-\nu)/(1+\nu)$
ω	Angular velocity
[ ]	Gives reference number on p.128-129
( )	Refers to equations given in the thesis

SYNOPSIS

This thesis deals with the problem of finding elastic field in a class of rotating bodies. The problems that have been attempted are mainly two-dimensional. The rotating bodies are composite and contain a region which is either an inhomogenity or an inclusion. For the inhomogenity, the elastic constants of the outer material are different from those of the insert. In the case of the inclusion, the elastic properties are the same as those of the outside material. The inclusion is a misfit, within elastic limits, in the outer material. Because of the inertia forces and the inhomogenity or the inclusion, the stresses develop in the composite material. The purpose of this thesis is to investigate some problems relating to rotating composite materials.

The same elastic problem may be seen from a different point of view, although they are mathematically similar. Consider a finite elastic body containing a hole in which either an inhomogenity or an inclusion is inserted. In the case of inhomogenity, we suppose that it is of the same shape and size as the hole and no relative slipping takes place. In the case of inclusion, its dimensions are slightly larger than those of the hole in the material. This may be realised as follows. First the inclusion is reduced to the size of the hole; it is then put in the hole, welded and left free. In the absence of the outside material,

the inclusion would have regained its original size. But because of the presence of the outside material, the stresses develop both in the insert and outside material. The composite body, containing the inhomogenity or the inclusion, is rotating about an axis which is perpendicular to the plane of the two-dimensional body, with a constant angular velocity.

The problems have been done as follows. First the stress field due to the external inertia forces is found everywhere in the medium. This is actually a particular integral of the stress equilibrium equations with the body forces and of the compatability equations written in terms of stresses and body force. This particular integral is a solution of the non-homogeneous equations. Secondly, we find the solution of these equations with body force set equal to zero. This solution of the homogeneous equations involves unknown constants or functions. The two solutions thus obtained are then added to arrive at the general solution of the equations. The constants or the unknown functions are determined by the given boundary conditions. It may be noted that in the problems attempted in this thesis, there are two sets of boundary conditions, one relating to the outer boundary and another at the equilibrium interface between the outer region and the inhomogenity or the inclusion. No external forces are acting along the outer boundary. At the equilibrium interface, the normal and shearing stresses are continuous and the continuity of the material is maintained.

For some problems, the classical Fourier series method and the integro-differential equation method suffice. Other problems may be more easily solved by the method of Hilbert problem. All these methods for the solution of the problems have been used to get explicit solutions.

In the first chapter, the basic equations of elasticity are given very briefly so as to make the thesis self-contained. These basic equations for three-dimensional case are reduced to the case of two-dimensional plane problems. In this case also, the equations of equilibrium and compatability equations are non-homogeneous because of the presence of body force. A particular integral of these equations are given in this chapter. A remark is made about the boundary conditions, when the body forces are present. The working of the problems starts from chapter 2.

In chapter 2, the problem of a rotating disc having a concentric circular inhomogenity is considered. The solution is obtained by finding two sets of complex potential functions : one for the inhomogenity and the other for the shell. These complex potentials are expanded into power series and the coefficients are determined by the conditions at the outer boundary of the shell and the continuity conditions at the equilibrium interface between the inhomogenity and the shell. Once the analytic functions are known, the stresses and displacements may be found directly for the shell and the inhomogenity.

In chapter 3, the problem of a rotating disc with an eccentric circular insert is considered. The region occupied by the composite rotating disc is conformally mapped to a circular ring. The problem is solved in the transformed region by finding two sets of complex potential functions. These analytic functions are determined by the series method as in the last chapter. The actual solution is obtained by transforming the solution back to the original plane by using the inverse mapping function. Some numerical work is given to find the actual stresses at the equilibrium interface between the insert and the shell.

In chapter 4, the problem of a rotating ring having an eccentric circular inclusion is considered. The problem is solved by finding two sets of complex potentials ; one for the insert and the other for the shell, using the theory of Hilbert problem. Some results pertaining to Hilbert problem are given in this chapter. The complex potentials were expanded in power series and the coefficients are evaluated using the boundary conditions. Numerical work is carried out to find the actual values of stresses in some typical examples.

In chapter 5, the problem of a rotating circular disc having two eccentric circular inclusions is considered. In this problem, three sets of complex potentials are obtained : one for the shell and others for the inserts. In evaluating the analytic

functions, the integro-differential equations are solved. Some results connected with integro-differential equations are given. The elastic field is calculated explicitely with the help of complex potentials.

In chapter 6, the problem of a rotating circular disc having an eccentric circular inclusion is considered. The theory of Hilbert problem and the integro-differential equation technique are used to find the complex potentials for the insert and the shell separately. For evaluating the integrals along the elliptic contour, the Cauchy integral formula for the infinite region is used.

In chapter 7 and 8, the problem of a rotating sphere having a concentric spherical inhomogenity is considered. The solution is obtained in two parts. The body force is written as the gradient of a potential ϕ, where ϕ is the sum of the potentials ϕ_1 and ϕ_2. The first part of the problem is the one where the body force is given by $\nabla\phi_1$, which happens to be purely a radial force. The solution is obtained by solving the equilibrium equations in spherical polar coordinates. The second part of the problem is the one when the body force is given by $\nabla\phi_2$, where ϕ_2 is a spherical solid harmonic. The solution is obtained by solving the Navier Stokes equations which may be derived from the equations given in chapter 1, using the theory of spherical

harmonics. The final solution is obtained by superposing the two solutions. It happens that the solution satisfies all the equations of the elasticity and the appropriate boundary conditions. By the uniqueness theorem, this solution is the solution of the problem. Numerical work is carried out to find the stresses in the insert, the shell and at the equilibrium boundary.

CHAPTER 1

Basic Equations

In classical theory of elasticity, the main problem is to find state of stress and strain in a body under certain prescribed boundary conditions. These conditions relate to either i) the external tractions or ii) the displacements or iii) the tractions at some part of the body and displacements at the remaining part of the body.

The state of stress as shown by A. Cauchy [1] at a point can be determined if the nine stress components P_{xx}, P_{yy}, P_{zz}, P_{xy}, P_{yx}, P_{xz}, P_{zx}, P_{zy}, P_{yz} are known; P_{ij} is the stress component on an element whose normal is in the direction of i and the component of the stress vector is in the jth direction. These stress components are related by the stress equilibrium equations

$$\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{yx}}{\partial y} + \frac{\partial P_{zx}}{\partial z} + F_x = 0 ,$$

$$\frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y} + \frac{\partial P_{zy}}{\partial z} + F_y = 0 ,$$

$$\frac{\partial P_{xz}}{\partial x} + \frac{\partial P_{yz}}{\partial y} + \frac{\partial P_{zz}}{\partial z} + F_z = 0 \tag{1.1}$$

where (F_x, F_y, F_z) are the components of the body force per unit volume and

$$P_{xy} = P_{yx}\ , \ P_{yz} = P_{zy}\ , \ P_{zx} = P_{xz}\ . \tag{1.2}$$

These equations (1.1) and (1.2) may be deduced from the consideration of the equilibrium of an element in the deformed state. If there is an element, the normal of which is in the direction of $\hat{n}$, the components of the stress vector P_{xx} , P_{xy} , P_{xz} are related to P_{ij} ($i,j = x,y,z$) by the relations

$$P_{nx} = P_{xx}l + P_{yx}m + P_{zx}n\ ,$$

$$P_{ny} = P_{xy}l + P_{yy}m + P_{zy}n\ ,$$

$$P_{nz} = P_{xz}l + P_{yz}m + P_{zz}n \tag{1.3}$$

where l, m, n are the direction cosines of the outward normal $\hat{n}$.

On the other hand, if we see the deformation of the body, it is well known that u, v, w are the displacement components at (x, y, z) in the X, Y, Z directions, then the state of deformation may be characterised by the six components e_{xx}, e_{yy}, e_{zz}, e_{xy}, e_{yz}, e_{zx} which are related to u, v, w by the relations

$$e_{xx} = \frac{\partial u}{\partial x}\ , \qquad e_{yy} = \frac{\partial v}{\partial y}\ , \qquad e_{zz} = \frac{\partial w}{\partial z}\ ,$$

$$e_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\ , \qquad e_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\ , \qquad e_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\ . \tag{1.4}$$

These will be termed as strain-displacement relations. The strains are related to the stresses by the generalised Hooke's law. In the

case of an isotropic elastic body, Hooke's law states that

$$P_{xx} = \lambda\theta + 2\mu\, e_{xx}\,, \qquad P_{xy} = \mu\, e_{xy}\,,$$

$$P_{yy} = \lambda\,\theta + 2\mu\, e_{yy}\,, \qquad P_{yz} = \mu\, e_{yz}\,,$$

$$P_{zz} = \lambda\,\theta + 2\mu\, e_{zz}\,, \qquad P_{zx} = \mu\, e_{zx} \tag{1.5}$$

where $\theta = e_{xx} + e_{yy} + e_{zz}$ and λ, μ are Lam's constants. Thus we have 15 equations (1.1, 1.4, 1.5) (where $P_{ij} = P_{ji}$) in 15 unknowns : six P_{ij}, six e_{ij}, and the three displacement components u, v, w. These equations are solved subject to the given boundary conditions. Note that the strains cannot be prescribed arbitrarily because they are related to the three variables u, v, w ; if (u,v,w) are singlevalued and are of class C_3, then

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \partial y}\,,$$

$$\frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} = \frac{\partial^2 e_{yz}}{\partial y\, \partial z}\,,$$

$$\frac{\partial^2 e_{zz}}{\partial x^2} + \frac{\partial^2 e_{xx}}{\partial z^2} = \frac{\partial^2 e_{zx}}{\partial z\, \partial x}\,,$$

$$\frac{\partial}{\partial z}\left(\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} - \frac{\partial e_{xy}}{\partial z} \right) = 2\, \frac{\partial^2 e_{zz}}{\partial x\, \partial y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}-\frac{\partial e_{yz}}{\partial x}\right) = 2\,\frac{\partial^2 e_{xx}}{\partial y\,\partial z},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial e_{xy}}{\partial z}+\frac{\partial e_{yz}}{\partial x}-\frac{\partial e_{zx}}{\partial y}\right) = 2\,\frac{\partial^2 e_{yy}}{\partial z\,\partial x}. \tag{1.6}$$

Only a few problems in the three-dimensional case have been solved. A great simplification is possible for the two dimensional problems. These are i) Plane strain problems and ii) plane stress problems. In Plane strain problems u and v are functions of (x,y) only and $w = 0$. This case arises in the study of deformations of large cylindrical bodies acted on by the external forces so distributed that the components of deformation in the direction of the axis of the cylinder vanishes and the remaining components do not vary along the length of the cylinder. Since $u = u(x,y)$ and $v = v(x,y)$, it atonce becomes apparent from (1.4), that the non-vanishing strain components are e_{xx}, e_{xy}, e_{yy} and non-zero stress components are P_{xx}, P_{xy}, P_{yy}, P_{zz}. But P_{zz} is related to P_{xx}, P_{yy} by the relation

$$P_{zz} = \nu(P_{xx} + P_{yy})$$

where ν is the Poisson's ratio. It follows that the equilibrium equations reduce to

$$\frac{\partial}{\partial x}\left(\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}-\frac{\partial e_{yz}}{\partial x}\right) = 2\,\frac{\partial^2 e_{xx}}{\partial y\,\partial z},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial e_{xy}}{\partial z}+\frac{\partial e_{yz}}{\partial x}-\frac{\partial e_{zx}}{\partial y}\right) = 2\,\frac{\partial^2 e_{yy}}{\partial z\,\partial x}. \tag{1.6}$$

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$$\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{yx}}{\partial y} + F_x = 0 ,$$

$$\frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y} + F_y = 0 \tag{1.7}$$

where F_x and F_y are functions of (x,y) only. The stress-strain relations become

$$P_{xx} = \lambda (e_{xx} + e_{yy}) + 2\mu e_{xx} ,$$

$$P_{yy} = \lambda (e_{xx} + e_{yy}) + 2\mu e_{yy} ,$$

$$P_{zz} = \lambda (e_{xx} + e_{yy}) ,$$

$$P_{xy} = \mu e_{xy} , \quad P_{yz} = P_{zx} = 0 . \tag{1.8}$$

If the Z-axis is taken in the direction of the axis of the cylinder, the boundary conditions on the curved surface reduce to

$$P_{nx} = P_{xx}l + P_{yx}m ,$$

$$P_{ny} = P_{xy}l + P_{yy}m \tag{1.9}$$

and the compatability equations (1.6) reduce to only one equation:

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \, \partial y} .$$

when the strain components are replaced by the stress components, the above equation may be written as

$$\nabla^2(P_{xx} + P_{yy}) = -\frac{2(\lambda+\mu)}{(\lambda+2\mu)} \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \right). \tag{1.10}$$

In this thesis, we shall attempt to solve mainly the Plane stress problems. For completeness sake, however, we mention that in the case of Plane stress we suppose

$$P_{xz} = P_{yz} = P_{zz} = 0.$$

This is applicable to the case of thin plates when the tractions are applied at the curved edges. This, however, does not simplify the problem as the non-zero stress and strain components are functions of z also. The complexity of the problem is reduced if the suggestion made by Filon [2] is followed. In this case for thin plates, we find the average values of the displacements and stresses

$$\bar{u}(x,y) = \frac{1}{2h} \int_{-h}^{h} u(x, y, z)\, dz ,$$

$$\bar{v}(x,y) = \frac{1}{2h} \int_{-h}^{h} v(x, y, z)\, dz ,$$

$$\bar{w}(x,y) = \frac{1}{2h} \int_{-h}^{h} w(x, y, z)\, dz ,$$

$$\bar{P}_{ij} = \frac{1}{2h} \int_{-h}^{h} P_{ij}(x, y, z)\, dz,$$

$$\bar{F}_{\alpha} = \frac{1}{2h} \int_{-h}^{h} F_{\alpha}(x, y, z)\, dz .$$

We assume that P_{xz} and P_{yz} are each zero on the faces of thin

plates and P_{zz} is zero throughout the thickness of the plate. In this case, as shown by Filon, the solution of the elastic problem is the same as in the case of Plane strain, with λ replaced by $\bar{\lambda} = 2\lambda\mu/(\lambda + 2\mu)$ and μ remaining unchanged and in place of the actual stresses and displacements, we find the average stresses and displacements. This assumption is valid because in the case of a thin plate, the stresses cannot vary very greatly along the thickness. This state of stress has been termed by Filon, as generalised Plane stress.

The systematic use of the complex variable theory for the solution of Plane strain problems was first proposed by Kolosseff [3] as early as 1909. This technique was fully exploited by a group of Russian mathematicians inspired by the work of Muskhelishvili [4] in U.S.S.R. The work remained unknown outside U.S.S.R. It was rediscovered by A.C. Stevenson [5] in U.K. A detailed study of these problems is given in Muskhelishvili's book [6] and in theoretical elasticity [7] by A.E. Green and W. Zerna and others [8,9].

In the absence of the body forces the solution to the problems of the plane theory of elasticity are considerably simplified. However, the non-homogeneous problem can always be reduced to the homogeneous case, by finding one of the many particular integrals. Thus, let $P_{\alpha\beta}^{(1)}$ $(\alpha,\beta = x,y)$ be any one such set of functions satisfying the equilibrium equations (1.7) and let the functions $P_{\alpha\beta}^{(2)}$ satisfy the equilibrium equations in the homogeneous case, namely

$$\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{yx}}{\partial y} = 0 ,$$

$$\frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y} = 0 . \tag{1.11}$$

Note that $P^{(2)}_{\alpha\beta}$ will contain requisite unknowns and the stress $P_{\alpha\beta} = P^{(1)}_{\alpha\beta} + P^{(2)}_{\alpha\beta}$ has to satisfy the given boundary conditions. It may be emphasized that $P_{\alpha\beta}$ are the actual stresses in the non homogeneous problem. The following remarks will be made regarding the boundary conditions. Let P_{nx} , P_{ny} be the given boundary tractions and let

$$P^{(1)}_{nx} = P^{(1)}_{xx} l + P^{(1)}_{yx} m ,$$

$$P^{(1)}_{ny} = P^{(1)}_{xy} l + P^{(1)}_{yy} m . \tag{1.12}$$

The stresses $P^{(2)}_{\alpha\beta}$ $(\alpha,\beta = x,y)$ are related to $P_{n\alpha}$ by

$$P^{(2)}_{n\alpha} = P_{n\alpha} - P^{(1)}_{n\alpha} \tag{1.13}$$

where $P^{(1)}_{n\alpha}$ are given by (1.12) and $P_{n\alpha}$ $(\alpha = x,y)$ are the actual tractions given at the boundary.

We shall be mainly dealing with the rotating bodies. In this case, the centrifugal force will be acting and its components shall be

$$F_x = \rho \omega^2 x, \quad F_y = \rho \omega^2 y, \quad F_z = 0 . \tag{1.14}$$

It may be seen that the equations (1.14) and the following stresses

$$p_{xx}^{(1)} = - \frac{\rho \omega^2}{8} (1 + 3\nu)(x^2 + y^2) - \frac{(1 - \nu)}{4} \rho \omega^2 x^2 ,$$

$$p_{yy}^{(1)} = - \frac{\rho \omega^2}{8} (1 + 3\nu)(x^2 + y^2) - \frac{(1 - \nu)}{4} \rho \omega^2 y^2 ,$$

$$p_{xy}^{(1)} = - \frac{\rho \omega^2}{4} (1 - \nu) \; xy \tag{1.15}$$

satisfy the equations (1.7). It may atonce be seen that apart from a rigid body displacement, the displacement components in this case are

$$u_x^{(1)} = - \frac{\rho \omega^2}{8} \frac{(1 - \nu^2)}{E} (x^2 + y^2) \; x ,$$

$$u_y^{(1)} = - \frac{\rho \omega^2}{8} \frac{(1 - \nu^2)}{E} (x^2 + y^2) \; y . \tag{1.16}$$

For the solution of the homogeneous boundary value problem one has to solve the equation (1.11) and the compatability equation

$$\nabla^2 (P_{xx} + P_{yy}) = 0 , \tag{1.17}$$

which is obtained from (1.10) by omitting the terms containing F_x F_y, subject to the modified boundary conditions which in terms of tractions shall be given by $P_{nx}^{(2)} + P_{ny}^{(2)}$, where

$$p_{nx}^{(2)} = p_{xx}^{(2)} l + p_{yx}^{(2)} m ,$$

$$p_{ny}^{(2)} = p_{xy}^{(2)} l + p_{yy}^{(2)} m . \tag{1.18}$$

Note that $p_{nx}^{(2)}$, $p_{ny}^{(2)}$ are given by (1.13).

In the absence of the body forces, these equations (1.11) and (1.17) are solved by the introduction of Airy's stress function $U(x,y)$ such that

$$P_{xx} = \frac{\partial^2 U}{\partial y^2} , \quad P_{xy} = - \frac{\partial^2 U}{\partial x \, \partial y} ; \quad P_{yy} = \frac{\partial^2 U}{\partial x^2} .$$

These satisfy the equation (1.11) and the equation (1.17) reduces to the biharmonic equation $\nabla^4 U = 0$. As shown in [6] , the solution of the biharmonic equation depends upon finding out two analytic functions $\phi(z)$ and $\psi(z)$. The stresses and displacements are related to the analytic functions $\phi(z)$ and $\psi(z)$ by the equations

$$P_{xx} + P_{yy} = 4 \text{ Re } \phi'(z) ,$$

$$P_{yy} - P_{xx} + 2 iP_{xy} = 2 [\bar{z} \phi''(z) + \psi'(z)] ,$$

$$2\mu (u + iv) = k \phi(z) - z \overline{\phi'(z)} - \overline{\psi(z)} . \tag{1.19}$$

It may also be noted that if

$$f_1 + i f_2 = i \int^s (P_{nx} + i P_{ny}) ds \tag{1.20}$$

then

$$\phi(z) + z\,\overline{\phi'(z)} + \overline{\psi(z)} = f_1 + i f_2 \,. \tag{1.21}$$

This is an important result and is used to find the functions $\phi(z)$ and $\psi(z)$ when the tractions are given on the boundary.

As is shown by Muskhelishvili, Green and Zerna and others in their books, if the region occupied by the body is simply connected, then $\phi(z)$ and $\psi(z)$ are single-valued analytic functions of z and they may be expanded by power series :

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \psi(z) = \sum_{n=0}^{\infty} b_n z^n \,.$$

If the region R is finitely multiply connected domain bounded by the exterior contour C_{m+1} and by the m interior contours C_K ($K = 1,2,\ldots, m$), then $\phi(z)$, $\psi(z)$ need not be single-valued. In that case, if it is assumed that the stress and displacements are single-valued, then $\phi(z)$ and $\psi(z)$ have the forms

$$\phi(z) = -\frac{1}{2\pi\,(1+k)} \sum_{K=1}^{m} \left\{X_1^{(K)} + i\,X_2^{(K)}\right\} \log(z-z_K) + \phi_0(z) \,,$$

$$\psi(z) = \frac{k}{2\pi\,(1+k)} \sum_{K=1}^{m} \left\{X_1^{(K)} - i\,X_2^{(K)}\right\} \log(z-z_K) + \psi_0(z) \tag{1.22}$$

where $(X_1^{(K)}, X_2^{(K)})$ is the resultant vector of the external forces applied to the contour C_K and z_K is an arbitrary point in the simply connected region R_K bounded by C_K. The functions $\phi_0(z)$ and $\psi_0(z)$ are single-valued analytic functions in R.

If R is an infinite region bounded by a simple closed contour C_1 with the origin lying inside and if the stress components are bounded in the neighbourhood of the point at infinity, then

$$\phi(z) = - \frac{X_1 + i X_2}{2\pi (1 + k)} \log z + (B + i c) z + \phi_o(z) ,$$

$$\psi(z) = \frac{k (X_1 - i X_2)}{2\pi (1 + k)} \log z + (B' + i c') z + \psi_o(z) \tag{1.23}$$

where (X_1 , X_2) are the components of the resultant vector of all external forces acting on the boundary. The functions $\phi_o(z)$, $\psi_o(z)$ are single-valued and analytic in the region R including the point at infinity. The constants B, B', c' are related to the state of stress at infinity by

$$2B - B' = P_{xx}(\infty), \quad 2B + B' = P_{yy}(\infty), \quad c' = P_{xy}(\infty) .$$

The constant c has no effect on the state of stress and is related to the rigid body rotation ω by the formula

$$c = \frac{2\mu}{1 + k} \omega$$

where

$$\omega = \lim_{z \to \infty} \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \text{ at infinity.}$$

In chapter 3 and subsequent chapters the forms of $\phi(z)$ and $\psi(z)$ under translation of the origin are needed and the results

pertaining to them are briefly stated here. Let the origin be shifted to a point (x_o, y_o) without rotation of axes. Let (x,y) and (x_1, y_1) be the coordinates of the same point in the old and new systems and let

$$z = x + i\,y\,, \qquad z_1 = x_1 + i\,y_1\,.$$

Obviously $z = z_1 + z_o$, where $z_o = x_o + i\,y_o$. Denote by $\phi_1(z_1)$ and $\Psi_1(z_1)$, the functions playing in the new system the same part as $\phi(z)$ and $\Psi(z)$ in the old one. Since the stress components are not altered by translation, one has by the equation $(1.19)_1$

$$\mathrm{Re}\;\phi(z) = \mathrm{Re}\;\phi_1(z_1) = \mathrm{Re}\;\phi_1(z - z_o)$$

whence

$$\phi(z) = \phi_1(z - z_o)\,. \tag{1.24}$$

One might have added on the right hand side any purely imaginary constant which would have no influence on the distribution of stress. The second formula gives

$$\begin{aligned}\bar{z}\,\phi'(z) + \Psi(z) &= \bar{z}_1\,\phi_1'(z_1) + \Psi_1(z_1)\\ &= (\bar{z} - \bar{z}_o)\,\phi_1'(z - z_o) + \Psi_1(z - z_o)\\ &= \bar{z}\,\phi_1'(z - z_o) + \Psi_1(z - z_o) - \bar{z}_o\,\phi_1'(z-z_o)\end{aligned}$$

whence

$$\Psi(z) = \Psi_1(z - z_o) - \bar{z}_o\,\phi_1'(z - z_o)\,. \tag{1.25}$$

Integration of (1.24) and (1.25) with respect to z gives

$$\phi(z) = \phi_1(z - z_o) \quad \text{and} \quad \psi(z) = \psi_1(z - z_o) - \bar{z}_o \, \phi_1'(z - z_o) \tag{1.26}$$

where certain arbitrary constants which do not affect the stress distribution have been omitted. It is seen that the function $\psi(z)$ is not invariant for a translation of the origin, i.e., the values for the old coordinates are not obtained by simply replacing in $\psi_1(z_1)$ the variable z_1 by $(z - z_o)$. In contrast, the function $\phi(z)$ is invariant.

The problems in two-dimensional elasticity are usually solved by three methods which may be briefly termed as 1) Fourier series method 2) Integro differential equation method 3) Method of linear relationship as termed by Muskhelishvili or more popularly known as Hilbert problem. At the appropriate stage, we shall make remarks for each of the methods.

This chapter gives the basic results of the theory of elasticity, which shall be used subsequently. In the next chapter, the problem of a rotating disc with a concentric insert is considered.

CHAPTER 2

Composite Rotating Disc

In this chapter, the problem of a rotating composite disc is considered. A circular medium called the 'insert' is embedded in a circular disc, with a concentric hole. The insert undergoes a spontaneous homogeneous deformation without slipping, which in the absence of the shell would be a given prescribed displacement. The elastic properties and the densities of the insert may be different from those of the outer région, the 'shell'. A uniform normal pressure P is applied at the boundary of the shell. The composite disc rotates with a constant angular velocity. The problem is treated as one of generalised Plane stress and the complex variable technique is used to find the exact analytical solution.

Consider a circular disc with a concentric hole of radius a and outer radius b. In this hole, a circular elastic insert of radius a is embedded. The insert is rigidly attached to the shell, so that there is no relative slipping. The insert undergoes a spontaneous deformation which in the absence of the shell is given by $u = \delta x$, $v = \delta y$ where δ lies within the elastic limits. The composite disc is rotating with a constant angular velocity, say ω, about an axis through their common centre perpendicular to the plane of the disc. Let the elastic constants of the insert be E', ν' and those of the shell be E, ν . The densities of the insert and the shell

are ρ' and ρ respectively. Let the outer boundary of the shell be L_0 and the interface between the shell and the insert be L. A constant normal pressure P is applied at the outer boundary L_0 of the composite disc.

As stated in chapter 1, in the absence of body forces, the solution to any two-dimensional plane elasticity problem depends upon the complex functions $\phi(z)$ and $\psi(z)$ which give the stress and displacement field. If the body is rotating, the centrifugal force $F_x = \rho\,\omega^2 x$, $F_y = \rho\,\omega^2 y$ comes into play. The particular integrals $P_{xx}^{(1)}$, $P_{xy}^{(1)}$, $P_{yy}^{(1)}$ are given by (1.15) and they satisfy the equilibrium equations. These stresses $P_{xx}^{(1)}$, $P_{xy}^{(1)}$, $P_{yy}^{(1)}$ may be transformed to polar coordinates, $P_{rr}^{(1)}$, $P_{r\theta}^{(1)}$, $P_{\theta\theta}^{(1)}$ by the relations [10]

$$P_{rr} + P_{\theta\theta} = P_{xx} + P_{yy} ,$$

$$P_{\theta\theta} - P_{rr} + 2i\,P_{r\theta} = e^{2i\theta}\,[\,P_{yy} - P_{xx} + 2i\,P_{xy}\,].$$

Thus

$$P_{rr}^{(1)} = -\frac{\rho\,\omega^2}{8}\,r^2\,(3+\nu),$$

$$P_{\theta\theta}^{(1)} = -\frac{\rho\,\omega^2}{8}\,r^2\,(1+3\nu),$$

$$P_{r\theta}^{(1)} = 0 . \tag{2.1}$$

Similarly the displacements u_x, u_y are related to u_r, u_θ, the displacements along the r and θ directions, by the relations

$$u_r + i\,u_\theta = e^{-i\theta}(u + i\,v).$$

Noting that $u_x^{(1)}$, $u_y^{(1)}$ are given by (1.16), we obtain

$$u_r^{(1)} = -\frac{\rho\,\omega^2}{8E}\,r^3\,(1-\nu^2)\,, \qquad u_\theta^{(1)} = 0. \tag{2.2}$$

Now we find the stresses $p_{\alpha\beta}^{(2)}$ $(\alpha,\beta = x,y)$ which satisfy the homogeneous equations (1.11) and (1.17). In this case, the boundary tractions $P_{n\alpha}$ $(\alpha = x,y)$ defined by the equations (1.9) on C are replaced by the modified boundary tractions (1.18), say $P_{n\alpha}^{(2)}$ $(\alpha = x,y)$.

At this stage, it is necessary to distinguish between the outer boundary L_o and the inner boundary L of the complete disc. The inner boundary L of the shell is the interface between the shell and the insert. At the outer boundary L_o, the stresses $p_{\alpha\beta}^{(1)}$ give rise to the tractions $P_{n\alpha}^{(1)}$ $(\alpha = x,y)$ where

$$P_{nx}^{(1)} = p_{xx}^{(1)}\cos\theta + p_{xy}^{(1)}\sin\theta\,,$$

$$P_{ny}^{(1)} = p_{xy}^{(1)}\cos\theta + p_{yy}^{(1)}\sin\theta \tag{2.3}$$

where θ is the angle between the outward normal $\hat{n}$ and the X-axis. The modified boundary tractions $P_{n\alpha}^{(2)}$ $(\alpha = x,y)$ is related to $P_{n\alpha}^{(1)}$ by $P_{n\alpha} = P_{n\alpha}^{(1)} + P_{n\alpha}^{(2)}$ where $P_{n\alpha}$ are the actual components of tractions on the body. If the complex functions $\phi_s(z)$, $\psi_s(z)$ give the elastic field in the shell due to the tractions $P_{nx}^{(2)}$, $P_{ny}^{(2)}$ we shall have

$$\phi_s(\sigma) + \sigma\,\overline{\phi_s'(\sigma)} + \overline{\psi_s(\sigma)} = f_1^{(2)} + i\, f_2^{(2)}$$

$$= i \int^s (P_{nx}^{(2)} + i\, P_{ny}^{(2)})\, ds \tag{2.4}$$

where we have replaced z by σ in (1.21) to distinguish a boundary point from a point in the interior of L_o. We now substitute the values of $P_{\alpha\beta}^{(1)}$ $(\alpha,\beta = x,y)$ from (1.15) into (2.3) and get $P_{nx}^{(1)}, P_{ny}^{(1)}$. Also since the disc is under a uniform normal pressure P, we get $P_{nx} = -P\cos\theta$, $P_{ny} = -P\sin\theta$. We obtain $P_{nx}^{(2)}$, $P_{ny}^{(2)}$ from (1.18) after substituting the values of $P_{\alpha\beta}$ and $P_{\alpha\beta}^{(1)}$. These values of $P_{nx}^{(2)}$ and $P_{ny}^{(2)}$ are substituted in the integral (2.4) and the integral is evaluated. We get

$$\phi_s(\sigma) + \sigma\,\overline{\phi_s'(\sigma)} + \overline{\psi_s(\sigma)} = \left[\frac{\rho\,\omega^2}{8}\, b^2(3+\nu) - P\right]\sigma\,. \tag{2.5}$$

This is the condition which is satisfied by $\phi_s(\sigma)$, $\psi_s(\sigma)$ at the outer boundary L_o.

At the boundary L, two boundary conditions are satisfied. One is the continuity of the stresses P_{rr}, $P_{r\theta}$ and another is the continuity of the material. We shall discuss the continuity of the stress first. Since P_{rr} and $P_{r\theta}$ are continuous on L, we have the relation

$$(P_{rr} + i\,P_{r\theta})_i = (P_{rr} + i\,P_{r\theta})_s\,.$$

Now, note that

$$(P_{rr} + i\,P_{r\theta}) = (P_{rr}^{(1)} + P_{rr}^{(2)}) + i\,(P_{r\theta}^{(1)} + P_{r\theta}^{(2)})\,. \tag{2.6}$$

The value of $(P_{rr}^{(2)} + i\, P_{r\theta}^{(2)})$ for the insert is related to $[\phi_i(z), \psi_i(z)]$ and $(P_{rr}^{(2)} + i\, P_{r\theta}^{(2)})$ for the shell is related to $[\phi_s(z), \psi_s(z)]$. The values of $(P_{rr}^{(1)} + i\, P_{r\theta}^{(1)})$ for the insert and the shell are given by (2.1). Substituting these values of $(P_{rr}^{(1)} + i\, P_{r\theta}^{(1)})$ and $(P_{rr}^{(2)} + i\, P_{r\theta}^{(2)})$ for the insert and the shell in terms of $[\phi_i(z), \psi_i(z)]$ and $[\phi_s(z), \psi_s(z)]$, we get

$$\phi_i(\sigma) + \sigma\, \overline{\phi_i'(\sigma)} + \overline{\psi_i(\sigma)} - \frac{\varrho' \omega^2}{8} a^2 (3 + \nu')\sigma$$

$$= \phi_s(\sigma) + \sigma\, \overline{\phi_s'(\sigma)} + \overline{\psi_s(\sigma)} - \frac{\varrho\, \omega^2}{8} a^2 (3 + \nu)\sigma . \tag{2.7}$$

As stated in the preceeding paragraph, we have to make use of the physical fact that there is no discontinuity of the material in the composite disc. This implies that the sum of discontinuity in the displacement of the shell and the insert at L_o with appropriate sign is equal to the prescribed displacement of the insert. The displacement of the shell at the boundary due to $P_{xx}^{(1)}$, $P_{xy}^{(1)}$, $P_{yy}^{(1)}$ is

$$u_x^{(1)} + i\, u_y^{(1)} = - \frac{\varrho\, \omega^2}{8} \frac{(1 - \nu^2)}{E} r^2 z \tag{2.8}$$

and similarly for the insert. Now, note that

i) $$u_x + i\, u_y = (u_x^{(1)} + i\, u_y^{(1)}) + (u_x^{(2)} + i\, u_y^{(2)}) \tag{2.9}$$

ii) $$(u_x + i\, u_y)_s - (u_x + i\, u_y)_i = - \delta(x + i\, y)$$
$$= - \delta z = - \delta a\, e^{i\theta} \tag{2.10}$$

We now substitute the values $\left[u_x^{(1)} + i\, u_y^{(1)}\right]$ in (2.9), from (2.8) and also $(u_x^{(2)} + i\, u_y^{(2)})$ in (2.9), in terms of ϕ, ψ and get the values of $(u_x + i\, u_y)$ for the shell and the insert separately. After substituting them in (2.10), we obtain

$$\frac{(3 - \nu')}{E'}\, \phi_i(\sigma) - \frac{(1 + \nu')}{E'}\, \sigma\, \overline{\phi_i'(\sigma)} - \frac{(1 + \nu')}{E'}\, \overline{\psi_i(\sigma)} + \delta\, \sigma$$

$$- \frac{\rho'\, \omega^2}{8E'}\, a^2 (1 - \nu'^2)\sigma \;=\; \frac{(3 - \nu)}{E}\, \phi_s(\sigma) - \frac{(1+\nu)}{E}\, \sigma\, \overline{\phi_s'(\sigma)}$$

$$- \frac{(1+\nu)}{E}\, \overline{\psi_s(\sigma)} - \frac{\rho\, \omega^2}{8E}\, a^2 (1 - \nu^2)\sigma \;.$$

(2.11)

We now proceed to obtain $\left[\phi_i(z), \psi_i(z)\right]$ and $\left[\phi_s(z), \psi_s(z)\right]$ from (2.6, 2.7, 2.11). Note that $\left[\phi_i(z), \psi_i(z)\right]$ and $\left[\phi_s(z), \psi_s(z)\right]$ are analytic in the regions occupied by the insert and the shell respectively. We may thereafter write

$$\phi_i(z) = \sum_{n=0}^{\infty} A_n\, z^n \;, \qquad \psi_i(z) = \sum_{n=0}^{\infty} B_n\, z^n \;,$$

$$\phi_s(z) = \sum_{n=-\infty}^{\infty} a_n\, z^n \;, \qquad \psi_s(z) = \sum_{n=-\infty}^{\infty} b_n\, z^n \;.$$

As is well known, the constants in $\left[\phi(z), \psi(z)\right]$ do not affect the stresses. Substituting the above forms in (2.8) and on substituting $\sigma = b\, e^{i\theta}$, it follows that

$$\frac{a_{-n}}{b^n} + (n+2)\,\bar{a}_{n+2}\, b^{n+2} + \bar{b}_n\, a^n = 0\,, \quad (n \geq 1),$$

$$a_n\, b^n - (n-2)\,\frac{\bar{a}_{-(n-2)}}{b^{\,n-2}} + \frac{\bar{b}_{-n}}{b^n} = 0\,, \quad (n \geq 2),$$

$$a_1 b + \bar{a}_1 b + \frac{\bar{b}_{-1}}{b} = \left[\frac{3+\nu}{8}\,\rho\,\omega^2\, b^3 - Pb\right],$$

$$a_2 = 0\,. \tag{2.12}$$

Multiplying the equation (2.7) by $\frac{(1+\nu)}{E}$ and adding to (2.11) and on substituting for $\left[\phi_1(z),\ \psi_1(z)\right]$, $\left[\phi_s(z),\ \psi_s(z)\right]$ it may be seen that

$$\frac{4}{E}\, a\, a_1 - \frac{\rho\,\omega^2}{8E}\, a^3(1-\nu^2) - \frac{\rho\,\omega^2}{8E}\, a^3(1+\nu)(3+\nu)$$

$$= \left[\frac{(3-\nu')}{E'} + \frac{(1+\nu)}{E}\right]\left\{A_1\, a\right\} + \left\{\frac{1+\nu}{E} - \frac{1+\nu'}{E'}\right\}\bar{A}_1 a$$

$$+ \delta a - \frac{\rho'\,\omega^2}{8E'}\, a^3(1-\nu'^2) - \frac{\rho'\,\omega^2 a^3(3+\nu')(1+\nu)}{8E}\,,$$

$$a_n = \frac{E}{4}\left\{\frac{3-\nu'}{E'} + \frac{1+\nu}{E}\right\} A_n\,, \quad (n \geq 2)\,,$$

$$\frac{a_{-n}}{a^n} = \frac{E}{4}\left\{\frac{(1+\nu)}{E} - \frac{(1+\nu')}{E'}\right\}(n+2)\,\bar{A}_{n+2}\, a^{n+2} +$$

$$\frac{E}{4}\left\{\frac{1+\nu}{E} - \frac{1+\nu'}{E'}\right\}\bar{B}_n\, a^n\,, \quad (n \geq 1)\,. \tag{2.13}$$

Multiplying the equation (2.7) with $\frac{(3-\nu)}{E}$ and subtracting (2.11) from it, it may be directly seen that

$$\frac{4}{E}\,\bar{a}_1\,a + \frac{4}{E}\,\frac{\bar{b}_{-1}}{a} + \frac{\rho\,\omega^2}{8E}\,a^3\,(1-\nu^2) - \frac{\rho\,\omega^2}{8E}\,a^3(3-\nu^2)$$

$$= \left\{\frac{3-\nu}{E} - \frac{3-\nu'}{E'}\right\} A_1\,a + \left\{\frac{3-\nu}{E} + \frac{1+\nu'}{E'}\right\} \bar{A}_1\,a$$

$$- \delta a + \frac{\rho'\,\omega^2}{8E'}\,a^3(1-\nu'^2) - \frac{\rho'\,\omega^2}{8E}\,a^3\,(3+\nu')(3-\nu)$$

and

$$(n+2)\,\bar{a}_{n+2}\,a^{n+2} + \bar{b}_n\,a^n = \frac{E}{4}\left\{\frac{3-\nu}{E} + \frac{1+\nu'}{E'}\right\}\left\{(n+2)\,\bar{A}_{n+2}\,a^{n+2}\right.$$

$$\left. + \bar{B}_n\,a^n\right\}, \qquad (n \geq 1),$$

$$(n-2)\,\frac{\bar{a}_{-(n-2)}}{a^{n-2}} - \frac{\bar{b}_{-n}}{a^n} = \frac{E}{4}\left\{\frac{3-\nu'}{E'} - \frac{3-\nu}{E}\right\} A_n\,a^n, \quad (n \geq 3). \tag{2.14}$$

On solving the equations (2.12), (2.13), (2.14) the only non zero quantities are given by

$$A_1 = \frac{\left[\dfrac{\delta EE'(a^2-b^2)}{2} - Pb^2E' + \dfrac{\rho\,\omega^2}{8}\,E'\left\{b^4(3+\nu) - 2a^2b^2(1+\nu) - a^4(1-\nu)\right\} - \dfrac{\rho'\,\omega^2 a^2}{16}\left\{E(a^2-b^2)(1-\nu'^2) - E'\left\{b^2(1+\nu) + a^2(1-\nu)\right\}(3+\nu')\right\}\right]}{\left[E'b^2(1+\nu) + E'a^2(1-\nu) + (b^2-a^2)\,E\,(1-\nu')\right]}$$

$$a_1 = \frac{\left[\frac{\delta E E' a^2}{2} - \frac{P b^2}{2}\left\{E'(1+\nu) + E(1-\nu')\right\} + \frac{\varrho' \omega^2}{8} E(1-\nu') a^4 + \frac{\varrho \omega^2}{16}\left\{\left\{E' b^4 (1+\nu) + E b^4 (1-\nu') - E a^4 (1-\nu')\right\}(3+\nu) + E' a^4 (1-\nu^2)\right\}\right]}{\left[E' b^2 (1+\nu) + E' a^2 (1-\nu) + (b^2 - a^2) E (1-\nu')\right]},$$

$$b_{-1} = \frac{\left[-\delta E E' a^2 b^2 - P a^2 b^2 \left\{E'(1-\nu) - E(1-\nu')\right\} - \frac{\varrho' \omega^2}{4} E(1-\nu') a^4 b^2 - \frac{\varrho \omega^2}{8} a^2 b^2 \left\{E b^2 (3+\nu)(1-\nu') + E' a^2 (1-\nu^2) - E' b^2 (3+\nu)(1-\nu) - E a^2 (3+\nu)(1-\nu')\right\}\right]}{\left[E' b^2 (1+\nu) + E' a^2 (1-\nu) + (b^2 - a^2) E (1-\nu')\right]}.$$

(2.15)

The complex potentials $\left\{\phi_1(z), \psi_1(z)\right\}$, $\left\{\phi_2(z), \psi_2(z)\right\}$ are then given by

$$\phi_1(z) = A_1 z\,, \qquad \psi_1(z) = 0\,,$$

$$\phi_2(z) = a_1 z\,, \qquad \psi_2(z) = \frac{b_{-1}}{z}$$

where A_1, a_1, b_{-1} are given by (2.15). The stresses can be calculated everywhere by the formula $P_{\alpha\beta} = P^{(1)}_{\alpha\beta} + P^{(2)}_{\alpha\beta}$ where $P^{(1)}_{\alpha\beta}$ $(\alpha, \beta = r, \theta)$ given by (2.1) and $P^{(2)}_{\alpha\beta}$ $(\alpha, \beta = r, \theta)$ may be obtained from the formulae

$$p_{rr}^{(2)} + p_{\theta\theta}^{(2)} = 4 \operatorname{Re} \phi'(z),$$

$$p_{\theta\theta}^{(2)} - p_{rr}^{(2)} + 2i\, p_{r\theta}^{(2)} = 2e^{2i\theta} \left[\bar{z}\, \phi''(z) + \psi'(z) \right]. \tag{2.16}$$

The displacements are given by $u_\alpha = u_\alpha^{(1)} + u_\alpha^{(2)}$ where $u_\alpha^{(1)}$ ($\alpha = r,\theta$) is given by (2.2) and $u_\alpha^{(2)}$ may be obtained from

$$2\mu\, (u_r^{(2)} + i\, u_\theta^{(2)}) = e^{-i\theta} \left[k\, \phi(z) - z\, \overline{\phi'(z)} - \overline{\psi(z)} \right]. \tag{2.17}$$

After doing all the simplifications cited above the actual stresses in the shell, shall come out to be

$$p_{rr} = \frac{\begin{aligned}&\Big[\delta E E' a^2 (1 - \frac{b^2}{r^2}) - P b^2 \Big\{ E'(1+\nu) + \frac{E' a^2 (1-\nu)}{r^2} + E(1-\nu')(1 - \frac{a^2}{r^2}) \Big\} \\ &+ \frac{\rho' \omega^2}{4} E(1-\nu')\, a^4 (1 - \frac{b^2}{r^2}) + \frac{\rho\, \omega^2}{8} \Big\{ E' b^2 (3+\nu) \Big\{ (b^2 - r^2)(1+\nu) \\ &+ \frac{b^2 a^2 (1-\nu)}{r^2} \Big\} + E' a^2 (1-\nu) \Big\{ a^2 (1+\nu)(1 - \frac{b^2}{r^2}) - r^2 (3+\nu) \Big\} \\ &+ E b^2 (1-\nu')(3+\nu)(b^2 - r^2 - \frac{a^2 b^2}{r^2}) + E(3+\nu)(1-\nu')\, a^2 \, x \\ &\qquad (r^2 - a^2 + \frac{a^2 b^2}{r^2}) \Big\} \Big]\end{aligned}}{\left[E' b^2 (1+\nu) + E' a^2 (1-\nu) + (b^2 - a^2)\, E(1-\nu') \right]},$$

$$
P_{\theta\theta} = \frac{\begin{aligned}&\Big[\delta EE'a^2(1+\frac{b^2}{r^2}) - Pb^2\Big\{E'(1+\nu) - \frac{E'(1-\nu)a^2}{r^2} \\ &+ E(1-\nu')(1+\frac{a^2}{r^2})\Big\} + \frac{\rho'\omega^2}{4}E(1-\nu')\ a^4(1+\frac{b^2}{r^2}) \\ &+ \frac{\rho\,\omega^2}{8}\Big\{E'b^2\Big\{b^2(1+\nu)(3+\nu) - \frac{a^2b^2(1-\nu)(3+\nu)}{r^2} \\ &- r^2(1+3\nu)(1+\nu)\Big\} + E'a^2(1-\nu)\Big\{a^2(1+\nu) + \frac{a^2b^2(1+\nu)}{r^2} \\ &- r^2(1+3\nu)\Big\} + Eb^2(1-\nu')\Big\{b^2(3+\nu) + \frac{a^2b^2(3+\nu)}{r^2} - r^2(1+3\nu)\Big\} \\ &- Ea^2(1-\nu')\Big\{a^2(3+\nu) + \frac{a^2b^2(3+\nu)}{r^2} - r^2(1+3\nu)\Big\}\Big\}\Big]\end{aligned}}{\Big[E'b^2(1+\nu) + E'a^2(1-\nu) + E(b^2-a^2)(1-\nu')\Big]},
$$

$$
P_{r\theta} = 0.
$$

In the case of insert, they shall come out to be

$$
P_{rr} = \frac{\begin{aligned}&\Big[\delta EE'(a^2-b^2) - 2Pb^2E' + \frac{\rho\,\omega^2}{4}E'\Big\{b^4(3+\nu) - 2a^2b^2(1+\nu) \\ &- a^4(1-\nu)\Big\} - \frac{\rho'\omega^2}{8}\Big\{E'(3+\nu')(r^2-a^2)\Big\{b^2(1+\nu)+a^2(1-\nu)\Big\} \\ &+ E(1-\nu')(b^2-a^2)\ \Big\{r^2(3+\nu') - a^2(1+\nu')\Big\}\Big\}\Big]\end{aligned}}{\Big[E'b^2(1+\nu) + E'a^2(1-\nu) + E(b^2-a^2)(1-\nu')\Big]},
$$

$$
P_{r\theta} = 0,
$$

$$P_{\theta\theta} = \frac{\begin{aligned}&\Big[\delta EE'(a^2 - b^2) - 2Pb^2E' + \frac{\rho\,\omega^2}{4}E'\left\{b^4(3+\nu) - 2a^2b^2(1+\nu)\right. \\ &\left. - a^4(1-\nu)\right\} - \frac{\rho'\omega^2}{8}\Big\{E'\left\{b^2(1+\nu) + a^2(1-\nu)\right\}\left\{r^2(1+3\nu')\right. \\ &\left. - a^2(3+\nu')\right\} + E(b^2 - a^2)(1-\nu')\left\{r^2(1+3\nu') - a^2(1+\nu')\right\}\Big\}\Big]\end{aligned}}{\Big[E'b^2(1+\nu) + E'a^2(1-\nu) + E(b^2 - a^2)(1-\nu')\Big]} .$$

The jump in the hoop stress at the equilibrium boundary can be easily seen to be

$$P^s_{\theta\theta} - P^i_{\theta\theta} = \frac{\begin{aligned}&\Big[2\delta EE'b^2 + 2Pb^2\left\{E'(1-\nu) - E(1-\nu')\right\} + \frac{\rho'\omega^2}{4}a^2(1-\nu') \times \\ &\Big\{E\left\{a^2(1-\nu') + b^2(1+\nu')\right\} - E'\left\{b^2(1+\nu) + a^2(1-\nu)\right\}\Big\} \\ &+ \frac{\rho\,\omega^2}{4}\Big\{E'\left\{a^4(1+\nu^2) + 2a^2b^2(1-\nu^2) - b^4(1+\nu)(3+\nu)\right\} \\ &+ E(1-\nu')\left\{b^4(3+\nu) - 2a^2b^2(1+\nu) - a^4(1-\nu)\right\}\Big\}\Big]\end{aligned}}{\Big[E'b^2(1+\nu) + E'a^2(1-\nu) + (b^2 - a^2)E(1-\nu')\Big]} .$$

Some particular cases may be stated. i) When $\delta = 0$, we get the case of an inhomogenity in a shell. ii) When $E' = E$, $\nu' = \nu$, $\rho' = \rho$ and $\delta = 0$, we get the case of a rotating solid disc, subjected to a normal pressure at the outer boundary. iii) When $E' = E$, $\nu' = \nu$, $\rho' = \rho$, $\delta \neq 0$ and $P = 0$, we get the case of an oversize insert having the elastic properties as the shell. The actual stresses shall be the following in the case when the outer boundary is free from tractions.

For the insert

$$P_{rr} = \frac{\varrho\omega^2}{8}(3+\nu)(b^2 - r^2) + \frac{\delta E}{2}\,\frac{(a^2 - b^2)}{b^2} ,$$

$$P_{\theta\theta} = \frac{\varrho\omega^2}{8}\left\{(3+\nu)b^2 - r^2(1+3\nu)\right\} + \frac{\delta E}{2}\,\frac{(a^2 - b^2)}{b^2} ,$$

$$P_{r\theta} = 0$$

and for the shell

$$P_{rr} = \frac{\varrho\omega^2}{8}(b^2 - r^2)(3+\nu) + \frac{\delta E a^2}{2b^2}\left(1 - \frac{b^2}{r^2}\right) ,$$

$$P_{\theta\theta} = \frac{\varrho\omega^2}{8}\left\{b^2(3+\nu) - r^2(1+3\nu)\right\} + \frac{\delta E a^2}{2b^2}\left(1 + \frac{b^2}{r^2}\right) ,$$

$$P_{r\theta} = 0 .$$

(2.18)

CHAPTER 3

Rotating Disc With An Eccentric Circular Insert

In this chapter, the problem of a rotating disc with an eccentric circular insert is considered. At the outset, it may be mentioned that the elastic properties and the density of the insert are the same as those of the shell. Thus the results of the previous chapter - of a concentic insert in a shell - cannot be deduced as a particular case of those obtained in this chapter; because the problem done in chapter 2 has elastic properties different from those in the shell. In the absence of the shell, the radius of the insert would be slightly larger (within the elastic limits) than that of the hole. Physically, one might consider the case, when the insert is compressed to the size of the hole, inserted into it, and is then welded at its rim to avoid slipping. Because of the misfit in size, the stresses would develop. The composite disc is rotating about the axis, perpendicular to the plane of the disc, through the centre of that disc which has the hole. The problem is solved by finding two sets of complex potentials : one for the insert and the other for the shell. The outer boundary of the disc is supposed to be free from tractions.

Consider a circular disc of radius unity, with an eccentric hole of radius r'. The distance between the centres is c where

$e \leq 1-r'$. No generality is lost by supposing that the centre of the eccentric hole lies on the X-axis. We suppose this to be the case. Let the region occupied by the disc with the hole be referred to as 'shell' and that occupying the region of the hole, as 'insert'. Let the centre of the matrix be 0 and that of the hole $0'$; 0 is taken as the origin of the coordinate system. The boundary of the hole is denoted by L and the outer boundary of the shell by L_o. (Fig. 1, p.32). The radius of the insert in its initial size is $r'(1+\delta)$ (δ lies within the elastic limits) before it is inserted in the shell. The composite body is rotating with a constant angular velocity ω. Let the elastic constants of the disc be E, ν and the density be ρ. Let $\{\phi_s(z),\ \psi_s(z)\}$ be the two complex potentials giving the elastic field in the shell and $\{\phi_i(z),\ \psi_i(z)\}$ in the insert for the homogeneous problem.

As noted in the first two chapters one has to solve the equations (1.1) and (1.5) subject to the appropriate boundary conditions. Because of the presence of the inertia forces, the particular integrals in this case are also given by (1.15, 1.16). The notation and the procedure followed here is the same as in chapter 2. One has to solve the homogeneous problem with modified boundary conditions on L_o and L. Let $P^{(2)}_{\alpha\beta}$ $(\alpha,\beta = x,y)$ and $u^{(2)}$, $v^{(2)}$ be the stresses and the displacements in the homogeneous problem. As was stated on p. 18, $\{\phi_s(z),\ \psi_s(z)\}$ satisfy the following boundary condition on L_o.

$$\phi_s(\sigma) + \sigma\,\overline{\phi_s'(\sigma)} + \overline{\psi_s(\sigma)} = f_1^{(2)} + i\,f_2^{(2)} \qquad (3.1)$$

where σ is any point on L_o . Now,

$$f_1^{(2)} + i\, f_2^{(2)} = i \int^s \left\{ p_{nx}^{(2)} + i\, p_{ny}^{(2)} \right\} ds$$

$$= i \int^s \left\{ (P_{nx} - P_{nx}^{(1)}) + i(P_{ny} - P_{ny}^{(1)}) \right\} ds$$

where $P_{n\alpha}$ is the given stress vector on L_o and $P_{n\alpha}^{(1)}$ $(\alpha = x,y)$ is given by (1.12). Since the boundary L_o is free from tractions $P_{n\alpha} = 0$ $(\alpha = x,y)$. It is easy to see that

$$f_1^{(2)} + i\, f_2^{(2)} = \frac{\varrho\,\omega^2}{8}(3+\nu)\,\sigma$$

and (3.1) becomes

$$\phi_s(\sigma) + \sigma\,\overline{\phi_s'(\sigma)} + \overline{\psi_s(\sigma)} = \frac{\varrho\,\omega^2}{8}(3+\nu)\sigma \quad . \qquad (3.2)$$

If the actual displacements are (u,v), then $u = u^{(1)} + u^{(2)}$ and $v = v^{(1)} + v^{(2)}$. Considering the insert alone, the displacements on the boundary L are given by

$$k\;\phi_i(\sigma) - \sigma\,\overline{\phi_i'(\sigma)} - \overline{\psi_i(\sigma)} = \frac{E}{1+\nu}\left\{u^{(2)} - \delta(x-c) + i(v^{(2)} - \delta y)\right\}$$

$$= \frac{E}{1+\nu}\left\{u - u^{(1)} - \delta(x-c) + i(v - v^{(1)} - \delta y)\right\}$$

(3.3)

where $u^{(1)}$, $v^{(1)}$ are given by the equation (1.16).

Considering the shell alone

$$k\ \phi_s(\sigma) - \sigma\,\overline{\phi_s'(\sigma)} - \overline{\psi_s(\sigma)} = \frac{E}{1+\nu}\left\{u - u^{(1)} + i(v - v^{(1)}\right\}. \quad (3.4)$$

We now insert the value of $\frac{E}{1+\nu}\left\{u - u^{(1)} + i\,(v - v^{(1)})\right\}$ from (3.4) in (3.3) to get

$$k\ \phi_i(\sigma) - \sigma\,\overline{\phi_i'(\sigma)} - \overline{\psi_i(\sigma)} + \frac{E}{1+\nu}\,\delta(\sigma - c)$$

$$= k\ \phi_s(\sigma) - \sigma\,\overline{\phi_s'(\sigma)} - \overline{\psi_s(\sigma)}\ . \quad (3.5)$$

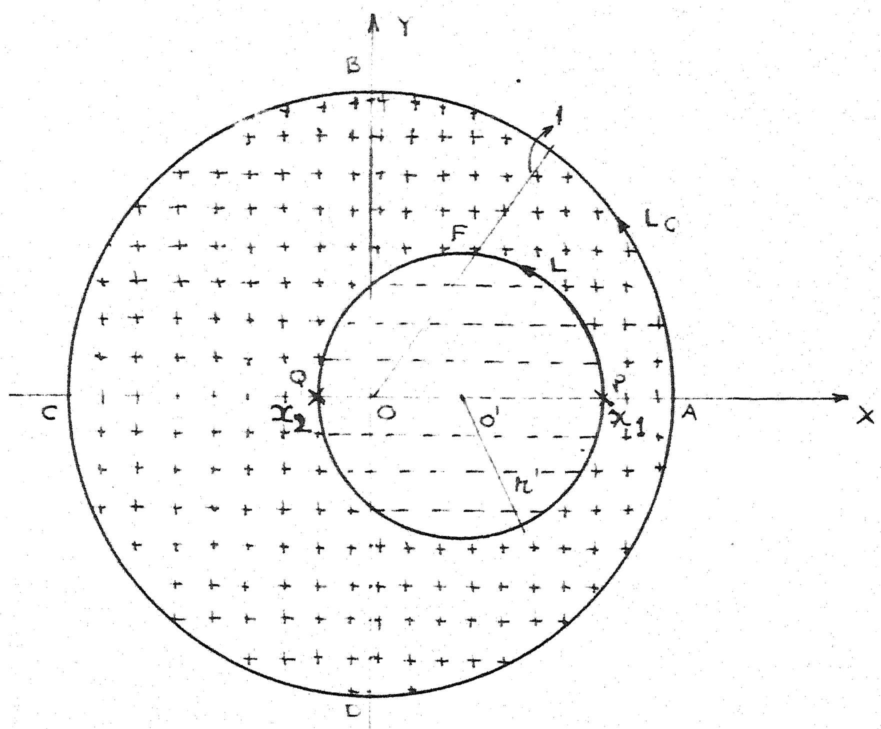
Also on L, the normal and shearing stresses are continuous

$$\phi_i(\sigma) + \sigma\,\overline{\phi_i'(\sigma)} + \overline{\psi_i(\sigma)} = \phi_s(\sigma) + \sigma\,\overline{\phi_s'(\sigma)} + \overline{\psi_s(\sigma)}. \quad (3.6)$$

We now introduce the mapping function $z = \frac{w - a}{aw - 1}$ [11] where we map the region in the Z-plane to a region in W-plane. The constant a is unknown as yet. Let the X-axis meet the boundary of the hole of the shell at the points P and Q (Fig.1, p.32) with the abscissa x_1 and x_2 respectively. Let $-1 < x_2 < x_1 < 1$, and

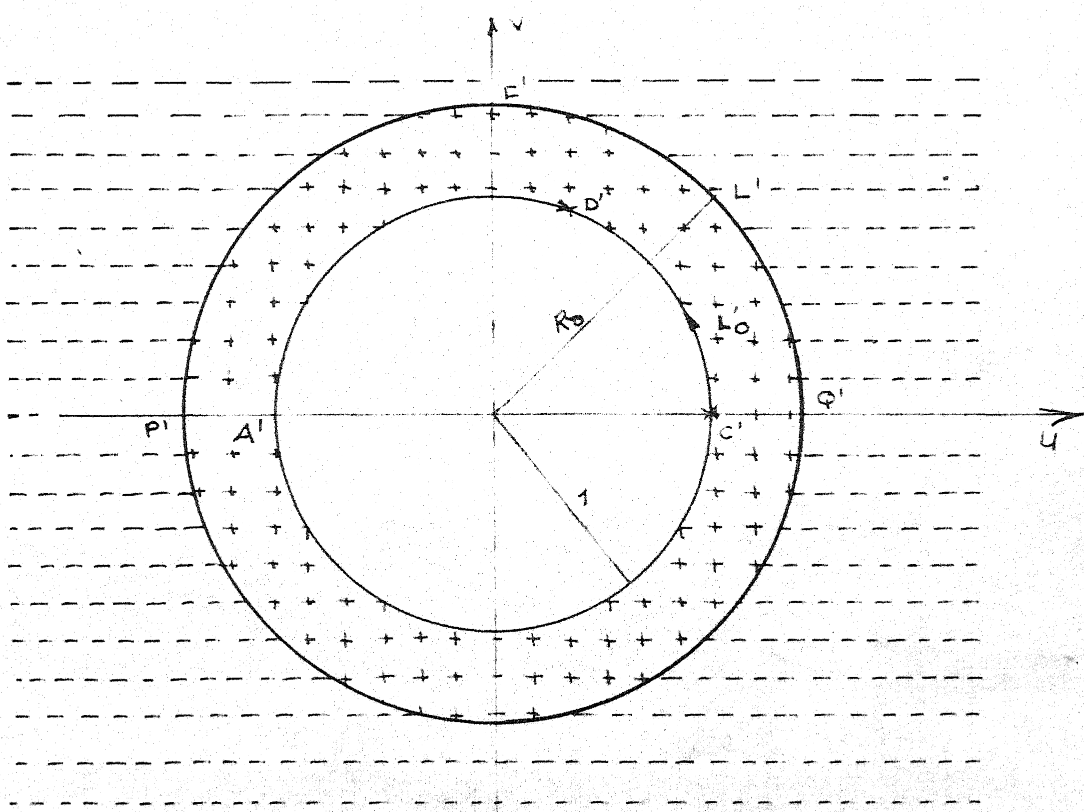
$$a = \frac{1 + x_1x_2 + \sqrt{(1 - x_1^2)(1 - x_2^2)}}{x_1 + x_2}\ .$$

Note that a is real and greater than 1. The mapping function transforms L_o in Z-plane to a circle L_o', with centre at the origin of the same radius in the W-plane; it transforms L to a concentric circle of radius R_o in the W-plane where $R_o > 1$ and



Z - PLANE

FIG. 1.



W - PLANE

FIG. 2.

$$R_o = \frac{1 - x_1 x_2 + \sqrt{(1 - x_1^2)(1 - x_2^2)}}{x_1 - x_2} .$$

The annular region between L_o and L is transformed to the region L_o' and L'. The region inside L is transformed to the region outside L' and the region outside L_o in the Z-plane goes to the region inside L_o' in the W-plane. The point at infinity in the W-plane corresponds to the point $\frac{1}{a}$ in the Z-plane. We now substitute the function $z = \frac{w - a}{aw - 1}$ in $\{\phi_s(z), \psi_s(z)\}$ and $\{\phi_i(z), \psi_i(z)\}$ and call the transformed functions $\{\phi_{so}(w), \psi_{so}(w)\}$ and $\{\phi_{io}(w), \psi_{io}(w)\}$. Note that the functions $\{\phi_{so}(w), \psi_{so}(w)\}$ are analytic in the annular region between L_o' and L' and $\{\phi_{io}(w), \psi_{io}(w)\}$ are analytic outside L' in the W-plane. Further we shall need the values of $\phi_{so}'(w)$ and $\psi_{so}'(w)$; where dash denotes the derivative with respect to the argument given in the bracket. It may be seen that

$$\phi_s'(z) = \frac{d}{dz} \phi_s(z) = \frac{d}{dw} \left\{ \phi_{so}(w) \right\} \frac{dw}{dz} = \frac{(aw-1)^2}{a^2-1} \phi_{so}'(w) .$$

The boundary conditions (3.2, 3.5, 3.6) in the Z-plane are transformed to the following in the W-plane.

$$\phi_{so}(t) + \frac{(t-a)(a\bar{t}-1)^2}{(at-1)(a^2-1)} \overline{\phi_{so}'(t)} + \overline{\psi_{so}(t)} = \frac{\rho\omega^2}{8} \frac{(3+\nu)(t-a)}{at-1} \tag{3.7}$$

$$k\ \phi_{so}(t) - \frac{(t-a)(a\bar{t}-1)^2}{(at-1)(a^2-1)}\ \overline{\phi'_{so}(t)} - \overline{\psi_{so}(t)} = k\ \phi_{1o}(t)$$

$$- \frac{(t-a)(a\bar{t}-1)^2}{(at-1)(a^2-1)}\ \overline{\phi'_{1o}(t)} - \overline{\psi_{1o}(t)} + \frac{E\delta}{1+\nu}\left\{\frac{(1-ac)\,t-(a-c)}{at-1}\right\}, \tag{3.8}$$

$$\phi_{so}(t) + \frac{(t-a)(a\bar{t}-1)^2}{(at-1)(a^2-1)}\ \overline{\phi'_{so}(t)} + \overline{\psi_{so}(t)} = \phi_{1o}(t)$$

$$+ \frac{(t-a)(a\bar{t}-1)^2}{(at-1)(a^2-1)}\ \overline{\phi'_{1o}(t)} + \overline{\psi_{1o}(t)} \tag{3.9}$$

where t is any point on either L' or L'_o .

Since $\phi_{so}(w)$, $\psi_{so}(w)$ are functions analytic in the finite region between L'_o and L', these have the forms [6]

$$\phi_{so}(w) = -\frac{(X_1 + iX_2)}{2\pi(1+k)}\ \log\,(w - w_1) + \phi^*_{so}(w)\ ,$$

$$\psi_{so}(w) = \frac{k(X_1 - iX_2)}{2\pi(1+k)}\ \log(w - w_1) + \psi^*_{so}(w)$$

where (X_1, X_2) is the resultant vector of the forces acting on L'_o and $\phi^*_{so}(w)$, $\psi^*_{so}(w)$ are single-valued functions analytic in the ring between L'_o and L'. Since no forces are acting on L'_o , it can be seen that

$$\phi_{so}(w) = \phi^*_{so}(w) = \sum_{n=-\infty}^{\infty} A_n w^n \quad \text{and} \quad \psi_{so}(w) = \psi^*_{so}(w) = \sum_{n=-\infty}^{\infty} B_n w^n \tag{3.10}$$

It may be remarked that A_o and B_o do not affect the stresses, whence one might take $A_o = B_o = 0$. Now

$$\phi_{so}(w) = \sum_{n=1}^{\infty} (A_{-n}w^{-n} + A_n w^n) \text{ and } \psi_{so}(w) = \sum_{n=1}^{\infty} (B_{-n}w^{-n} + B_n w^n). \tag{3.11}$$

The functions $\{\phi_{1o}(w), \psi_{1o}(w)\}$ are analytic in the infinite region bounded by L' and hence

$$\phi_{1o}(w) = -\frac{(X_1 + iX_2)}{2\pi(1+k)} \log w + \Gamma w + \phi^*_{1o}(w),$$

$$\psi_{1o}(w) = \frac{k(X_1 - iX_2)}{2\pi(1+k)} \log w + \Gamma' w + \psi^*_{1o}(w) \tag{3.12}$$

where the assumption is that the stresses are bounded at infinity. In expression (3.12), it may be taken that $X_1 = X_2 = 0$, $\Gamma = \Gamma' = 0$, which follow from the boundedness of the displacement at infinity and the stresses being zero at infinity. Then from (3.12) we obtain

$$\phi_{1o}(w) = \phi^*_{1o}(w) \quad ; \quad \psi_{1o}(w) = \psi^*_{1o}(w) .$$

Note that $\phi^*_{1o}(w)$, $\psi^*_{1o}(w)$ are single-valued analytic functions in the infinite region bounded by the circle L'_o. Leaving the constant terms which do not affect the stresses $\{\phi_{1o}(w), \psi_{1o}(w)\}$ may be taken as

$$\phi_{1o}(w) = \sum_{n=1}^{\infty} \frac{a_n}{w^n} \quad ; \quad \psi_{1o}(w) = \sum_{n=1}^{\infty} \frac{b_n}{w^n} . \tag{3.13}$$

Adding the boundary conditions (3.8) and (3.9), it is easy to get

$$\phi_{1o}(t) + \frac{E\delta}{4}\left\{\frac{t(1-ac)-(a-c)}{at-1}\right\} = \phi_{so}(t) \qquad (3.14)$$

where $t = R_o e^{i\theta}$. We substitute the values of $\phi_{1o}(t)$ and $\phi_{so}(t)$ from (3.13) and (3.11) and also put $t = R_o e^{i\theta}$ in (3.14). We then equate the like powers of θ. The following equations are then obtained.

$$aa_1 - aA_{-1} = \frac{E\delta}{4}(a - c) ,$$

$$A_1 = -\frac{E\delta}{4}(1 - ac) ,$$

$$A_n = a^{n-1} A_1 , \quad (n \geq 2) ,$$

$$aA_{-n} - A_{-(n-1)} = aa_n - a_{n-1} , \quad (n \geq 2) . \qquad (3.15)$$

Multiplying (3.9) with k and subtracting (3.8) from it, it can be seen that

$$\frac{(t-a)(a\bar{t}-1)^2}{(at-1)(a^2-1)} \overline{\phi'_{1o}(t)} + \overline{\psi_{1o}(t)} - \frac{E\delta}{4}\left\{\frac{t(1-ac)-(a-c)}{at-1}\right\}$$

$$= \frac{(t-a)(a\bar{t}-1)^2}{(at-1)(a^2-1)} \overline{\phi'_{so}(t)} + \overline{\psi_{so}(t)} .$$

Substituting for $\{\phi_{1o}(t), \psi_{1o}(t)\}$ and $\{\phi_{so}(t), \psi_{so}(t)\}$ the following equations are obtained.

$$\frac{a^3\,\bar{a}_1}{a^2-1} + \frac{E\Delta}{4}(a - c) = \frac{1}{a^2-1}\left\{2\bar{A}_2 R_o^2 - a(1 + 2R_o^2)\,\bar{A}_1 + a^3\,\bar{A}_{-1}\right\}$$

$$+ a\bar{B}_1\, R_o^2 \quad ,$$

$$\left\{\frac{2a^3\,\bar{a}_2 - a^2(2 + R_o^2)\,\bar{a}_1}{R_o(a^2-1)}\right\} - \frac{E\Delta}{4}(1 - ac)\;R_o - \frac{\bar{b}_1}{R_o}$$

$$= \frac{1}{a^2-1}\left\{\bar{A}_1 R_o - \frac{a^2(2 + R_o^2)}{R_o}\,\bar{A}_{-1} + \frac{2a^3\,\bar{A}_{-2}}{R_o}\right\} - \frac{\bar{B}_{-1}}{R_o} \quad ,$$

(3.16)

$$\frac{1}{a^2-1}\left\{\frac{-n\,\bar{a}_n}{R_o^n} + \frac{(n+1)\,a(1+2R_o^2)\;\bar{a}_{n+1}}{R_o^{\,n+2}} - \frac{(n+2)\,a^2(R_o^2 + 2)\;\bar{a}_{n+2}}{R_o^{\,n+2}}\right.$$

$$\left. + \frac{(n+3)\,a^3\,\bar{a}_{n+3}}{R_o^{\,n+2}}\right\} + \left(\frac{a\bar{b}_{n+1}}{R_o^n} - \frac{\bar{b}_{n+2}}{R_o^{n+2}}\right)$$

$$= \frac{1}{a^2-1}\left\{-\frac{n\bar{A}_{-n}}{R_o^n} + \frac{(n+1)\,a(1 + 2R_o^2)\;\bar{A}_{-(n+1)}}{R_o^{\,n+2}} - \frac{(n+2)\,a^2(R_o^2 + 2)\,\bar{A}_{-(n+2)}}{R_o^{\,n+2}}\right.$$

$$\left. + \frac{(n+3)\,a^3\,\bar{A}_{-(n+3)}}{R_o^{\,n+2}}\right\} + \left\{\frac{a\bar{B}_{-(n+1)}}{R_o^n} - \frac{\bar{B}_{-(n+2)}}{R_o^{\,n+2}}\right\}, \quad (n \geq 0)$$

and

$$\frac{1}{a^2-1}\left\{-a^3 n\bar{A}_n + (n+1)\,a^2(R_o^2 + 2)\,\bar{A}_{n+1} - (n+2)\,a(1+2R_o^2)\;\bar{A}_{n+2}\right.$$

$$\left. + (n+3)\,\bar{A}_{n+3}\;R_o^2\right\} + \left(a\bar{B}_{n+2}\;R_o^2 - \bar{B}_{n+1}\right) = 0 \;, \;(n \geq 0)\,. \quad (3.17)$$

L_0' , $t = e^{i\theta}$ and from equation (3.7), it may be seen that

$$a A_{-1} + \frac{1}{a^2 - 1}\left\{2\bar{A}_2 - 3a\bar{A}_1 + a^3\,\bar{A}_{-1}\right\} + a\bar{B}_1 = -\frac{\rho\,\omega^2}{8}(3+\nu)\,a \ ,$$

$$-A_1 + \frac{1}{a^2 - 1}\left\{\bar{A}_1 - 3a^2\,\bar{A}_{-1} + 2a^3\,\bar{A}_{-2}\right\} - \bar{B}_{-1} = \frac{\rho\,\omega^2}{8}(3+\nu) \ ,$$

(3.18)

$$(aA_{n+1} - A_{n+2}) + \frac{1}{a^2 - 1}\left\{-n\bar{A}_{-n} + 3(n+1)\,a\bar{A}_{-(n+1)} - 3(n+2)\,a^2\,\bar{A}_{-(n+2)}\right.$$

$$\left. + (n+3)\,a^3\,\bar{A}_{-(n+3)}\right\} + \left\{a\bar{B}_{-(n+1)} - \bar{B}_{-(n+2)}\right\} = 0 \ , \quad (n \geq 0) ,$$

$$(aA_{-(n+2)} - A_{-(n+1)}) + \frac{1}{a^2 - 1}\left\{-a^3 n\bar{A}_n + 3(n+1)\,a^2\,\bar{A}_{(n+1)}\right.$$

$$\left. - 3(n+2)\,a\bar{A}_{(n+2)} + (n+3)\,\bar{A}_{n+3}\right\} + (a\bar{B}_{n+2} - \bar{B}_{n+1}) = 0, \quad (n \geq 0) .$$

(3.19)

Solving the above equations for the unknown quantities, it is seen that all of them are real and are given by

$$A_1 = \frac{E\delta}{4}(ac - 1) \ ,$$

$$B_1 = \frac{E\delta}{4}\,\frac{(2a - c)}{aR_0^2} \ ,$$

$$a_1 = \frac{(a^2 - 1)}{a(2a^2 - 1)}\left\{\frac{E\delta}{4}\,\frac{(2a - c)(R_0^2 - 1)}{R_0^2} - \frac{\rho\,\omega^2}{8}\,a(3+\nu)\right\} ,$$

$$B_{-1} = \frac{E\delta}{4}\left\{\frac{(R_o^2-1)}{R_o^2}\;\frac{a(2a-c)}{1-2a^2} + \frac{(R_o^2-1)}{R_o^4}\;\frac{2a(2a-c)}{(a^2-1)} + (2-ac)\right\}$$

$$-\frac{\varrho\omega^2}{8}\;\frac{(3+\nu)(1-a^2)}{(1-2a^2)}\;,$$

$$b_1 = \frac{E\delta}{4}\left\{\frac{(R_o^2-1)}{R_o^2}\;\frac{a(2a-c)}{1-2a^2} + \frac{(R_o^2-1)\,2a(2a-c)}{R_o^4\,(a^2-1)} + (ac-2)(R_o^2-1)\right\}$$

$$-\frac{\varrho\omega^2}{8}\;\frac{(3+\nu)(1-a^2)}{(1-2a^2)}\;,$$

$$A_{-1} = \frac{a^2-1}{a(2a^2-1)}\left\{\frac{E\delta}{4}\;\frac{(2a-c)(R_o^2-1)}{R_o^2} - \frac{\varrho\omega^2}{8}\;a(3+\nu)\right\}$$

$$-\frac{E\delta}{4}\;\frac{(a-c)}{a}\;,$$

and

$$A_n = a^{n-1}A_1\;,\quad(n \geq 2)\;,$$

$$B_n = \frac{B_1}{a^{n-1}\,R_o^{2n-2}}\;,\quad(n \geq 2)\;,$$

$$A_{-n} = \frac{A_{-1}}{a^{n-1}} + \frac{R_o^{2n-2}-1}{a^{n-1}R_o^{2n-2}}\;B_1\;,\quad(n \geq 2)\;,$$

$$a_n = \frac{a_1}{a^{n-1}} + \frac{R_o^{2n-2}-1}{a^{n-1}\,R_o^{2n-2}}\;B_1\;,\quad(n \geq 2)\;,$$

$$B_{-(n+2)} = a^{n+1} B_{-1} + \frac{(R_o^2 - 1)^2}{R_o^4} \frac{B_1}{a^2 - 1} \left[-3a^{n+1} + \Delta a^{n-1} \left\{ \frac{a^2(R_o^2 - 4)}{a^2 R_o^2 - 1} \right.\right.$$

$$\left.\left. + \frac{(R_o^2 - 1)(a^{2n-2} R_o^{2n-2} - 1)}{(a^2 R_o^2 - 1)^2 a^{2n-4} R_o^{2n-4}} - \frac{(R_o^2 - 4) + (n-1) R_o^2 (R_o^2 - 1)}{(a^2 R_o^2 - 1) a^{2n-2} R_o^{2n}} \right\} \right],$$

$$(n \geq 0),$$

$$b_{n+2} = b_1 a^{n+1} R_o^{2n+2} - a^{n+1} \left\{ R_o^{2n+2} - 1 \right\} \bar{B}_{-1} + \frac{(R_o^2 - 1)^2}{R_o^4} \frac{B_1}{a^2 - 1} \times$$

$$\left[-3a^{n+1} + \Delta a^{n-1} \left\{ \frac{a^2(R_o^2 - 4)}{(a^2 R_o^2 - 1)} + \frac{(R_o^2 - 1)(a^{2n-2} R_o^{2n-2} - 1)}{(a^2 R_o^2 - 1)^2 a^{2n-4} R_o^{2n-4}} \right.\right.$$

$$\left.\left. - \frac{(R_o^2 - 4) + (n-1) R_o^2 (R_o^2 - 1)}{(a^2 R_o^2 - 1) a^{2n-2} R_o^{2n}} \right\} \right], \quad (n \geq 0)$$

$$(3.20)$$

where $\Delta = 0$ for $n = 0$ and $\Delta = 1$ for $n \geq 1$. Substituting the values of the constants in (3.11), (3.13) and substituting $w = \frac{z - a}{az - 1}$, we get the complex potential functions.

$$\phi_1(z) = \frac{a_1 a(1 - az)}{a^2 - 1} + \frac{B_1 a(R_o^2 - 1)(az - 1)^2}{(1 - a^2) \left\{ a(R_o^2 - 1)z + 1 - a^2 R_o^2 \right\}},$$

$$\phi_2(z) = \frac{a(1 - az)}{a^2 - 1} A_{-1} + \frac{B_1 a(R_o^2 - 1)(az - 1)^2}{(1 - a^2) \left\{ a(R_o^2 - 1)z + 1 - a^2 R_o^2 \right\}}$$

$$+ \frac{(z - a)}{a^2 - 1} A_1,$$

$$\psi_1(z) = \frac{b_1(az-1)}{\left\{(1-a^2R_o^2)z + a(R_o^2-1)\right\}} + \frac{a(R_o^2-1)(az-1)^2\,B_{-1}}{z(a^2-1)\left\{(1-a^2R_o^2)z+a(R_o^2-1)\right\}}$$

$$+ \frac{(R_o^2-1)^2}{R_o^4}\,\frac{B_1}{a^2-1}\left[\frac{3a(az-1)^2}{z(z-a)(a^2-1)} - \frac{a(az-1)^3}{(a^2-1)z(z-a)} \times\right.$$

$$\left.\frac{1}{\left\{a(R_o^2-1)z+(1-a^2R_o^2)\right\}}\left\{(R_o^2-4) + \frac{R_o^2(R_o^2-1)(az-1)}{a(R_o^2-1)z+(1-a^2R_o^2)}\right\}\right],$$

$$\psi_s(z) = \frac{(az-1)}{(1-a^2)z}B_{-1} + \frac{aR_o^2(z-a)}{(a^2R_o^2-1)z+a(1-R_o^2)}\,B_1 + \frac{(R_o^2-1)^2}{R_o^4\,(a^2-1)}\,B_1 \times$$

$$\left[\frac{3a(az-1)^2}{z(z-a)(a^2-1)} - \frac{a(az-1)^3}{z(z-a)(a^2-1)\left\{a(R_o^2-1)z+1-a^2R_o^2\right\}} \times\right.$$

$$\left.\left\{(R_o^2-4) + \frac{R_o^2(R_o^2-1)(az-1)}{\left\{a(R_o^2-1)z+1-a^2R_o^2\right\}}\right\}\right].$$

The stresses and the displacements, in the shell and in the insert, for the homogeneous problem may be calculated using the formulae (2.16) and (2.17). To get the stresses and displacements in the non-homogeneous problem, we have to add the stresses and displacements given by (1.15) and (1.16) respectively, to those obtained in the homogeneous problem. The jump in the hoop stress at the equilibrium boundary can be seen to be

$$P^{s}_{\theta\theta} - P^{i}_{\theta\theta} = \frac{E\delta}{2}\left[\, 1 + \frac{R_o^2(1-a^2)^2 \left\{(1-a^2R_o^2)^2\, r^2 + 2ar(R_o^2-1)(1-a^2R_o^2)\cos\theta + a^2(R_o^2-1)^2 \cos 2\theta\right\}}{\left\{(1-a^2R_o^2)^2\, r^2 + 2ar(1-a^2R_o^2)(R_o^2-1)\cos\theta + a^2(R_o^2-1)^2\right\}^2}\right.$$

The case of a concentric circular inclusion in the circular disc may be derived from the present one by taking the limit when a tends to infinity. This places the points x_1 and x_2 symmetrically on the X-axis with respect to the origin. In this case, the stresses are the following.

For the insert

$$P_{rr} = \frac{\delta E(r'^2 - 1)}{2} + \frac{\rho\,\omega^2}{8}(3+\nu)(1 - r^2)\ ,$$

$$P_{\theta\theta} = \frac{\delta E(r'^2 - 1)}{2} + \frac{\rho\,\omega^2}{8}\left\{(3+\nu) - r^2(1+3\nu)\right\},$$

$$P_{r\theta} = 0$$

and for the shell

$$P_{rr} = \frac{\delta E}{2}\, r'^2\left(1 - \frac{1}{r^2}\right) + \frac{\rho\,\omega^2}{8}(3+\nu)(1 - r^2)\ ,$$

$$P_{\theta\theta} = \frac{\delta E}{2}\, r'^2\left(1 + \frac{1}{r^2}\right) + \frac{\rho\,\omega^2}{8}\left\{(3+\nu) - r^2(1+3\nu)\right\},$$

$$P_{r\theta} = 0\ .$$

These stresses are the same as those given by (2.18), with $b = 1$.

Numerical work for the calculation of stresses both in the insert and in the shell has been done on Computer CDC 3600. The Poisson's ratio was taken as 1/3 and $\delta = .01$. The value of $\rho\,\omega^2/E$ was taken to be .005. The points x_1 and x_2 where the inner boundary of the shell cuts the X-axis were taken to be 0.8 and 0.4 and the distance between the centres was taken as e. Four cases have been discussed, with $e = 0.60$, 0.65, 0.70 and 0.75 respectively. The radius of the insert remains the same, 0.2, in each case. The normal, shearing stresses were found at the equilibrium boundary. They are given for the insert and the shell in Tables 3.1, 3.2, 3.3, 3.4 on pages 44, 45, 46, 47 respectively. The first table is for $e = 0.6$. The first column gives the angular distance θ in degrees with origin as O', the centre of the insert, measured from the X-axis in anti-clockwise direction. The second and third columns give the stresses P_{rr}/E, $P_{r\theta}/E$, which are the same (as physically should be) for the insert and the shell. The fourth and fifth columns give the hoop stress for the insert and the shell at the equilibrium boundary respectively. In tables 3.2, 3.3 and 3.4, the same results are given for $e = 0.65$, 0.70, 0.75 respectively.

TABLE 3.1

$e = 0.6$

θ	P_{rr}/E	$P_{r\theta}/E$	$P_{\theta\theta}/E$ Shell	$P_{\theta\theta}/E$ Insert
0	-0.00438	0.00000	0.00956	-0.00044
30	-0.00453	-0.00005	0.00943	-0.00057
60	-0.00487	0.00000	0.00935	-0.00065
90	-0.00500	0.00013	0.00953	-0.00067
120	-0.00505	0.00047	0.00947	-0.00053
150	-0.00430	0.00109	0.00885	-0.00115
180	-0.00310	0.00000	0.00772	-0.00228

TABLE 3.2

$c = 0.65$

θ	P_{rr}/E	$P_{r\theta}/E$	$P_{\theta\theta}/E$ Shell	$P_{\theta\theta}/E$ Insert
0	-0.00441	0.00000	0.01063	0.00063
30	-0.00476	-0.00029	-0.01033	0.00033
60	-0.00573	-0.00030	0.01032	0.00032
90	-0.00628	-0.00002	0.01044	0.00044
120	-0.00660	0.00093	0.01074	0.00074
150	-0.00492	0.00234	0.00917	-0.00083
180	-0.00246	0.00000	0.00678	-0.00323

TABLE 3.3

$e = 0.70$

θ	P_{rr}/E	$P_{r\theta}/E$	$P_{\theta\theta}/E$ Shell	$P_{\theta\theta}/E$ Insert
0	-0.00425	0.00000	0.01278	-0.00278
30	-0.00497	-0.00084	0.01178	0.00178
60	-0.00738	-0.00088	0.01186	0.00186
90	-0.00881	-0.00041	0.01227	0.00227
120	-0.00980	0.00192	0.01307	0.00307
150	-0.00604	0.00503	0.00939	-0.00061
180	-0.00093	0.00000	0.00432	-0.00568

TABLE 3.3

$e = 0.70$

θ	P_{rr}/E	$P_{r\theta}/E$	$P_{\theta\theta}/E$ Shell	$P_{\theta\theta}/E$ Insert
0	-0.00425	0.00000	0.01278	-0.00278
30	-0.00497	-0.00084	0.01178	0.00178
60	-0.00738	-0.00088	0.01186	0.00186
90	-0.00881	-0.00041	0.01227	0.00227
120	-0.00980	0.00192	0.01307	0.00307
150	-0.00604	0.00503	0.00939	-0.00061
180	-0.00093	0.00000	0.00432	-0.00568

TABLE 3.4

$e = 0.75$

θ	P_{rr}/E	$P_{r\theta}/E$	$P_{\theta\theta}/E$ Shell	$P_{\theta\theta}/E$ Insert
0	-0.00387	0.00000	0.01874	0.00874
30	-0.00558	-0.00260	0.01452	0.00452
60	-0.01154	-0.00236	0.01401	0.00401
90	-0.01606	-0.00243	0.01618	0.00618
120	-0.01971	0.00484	0.01932	0.00932
150	-0.00903	0.01365	0.00863	-0.00137
180	-0.00456	0.00000	-0.00492	-0.01492

CHAPTER 4

Rotating Ring With An Eccentric Circular Insert

Consider the case when a region occupied by the body is a circular ring bounded by two concentric circles and with its centre as the origin of coordinate system. Let an eccentric circular region in the ring (region A, in Fig. 3, p.49) undergo a spontaneous homogeneous and uniform deformation within elastic limits. The ring is then stressed. The same physical problem may be described by considering the following. Consider the circular ring as described above and let there be a circular hole in which another circular disc, the insert, of slightly larger radius is inserted. It may be seen that the two problems are mathematically similar. The composite material is rotating about an axis through the origin perpendicular to the plane of the disc. The problem is to find the elastic field everywhere in the medium.

As stated in chapter 1, any two-dimensional generalised plane stress problem is solved by finding a set of two complex potential functions $\phi(z)$, $\psi(z)$. In this case there will be two sets : one for the insert and another for the remaining part of the ring. The problem is done with the help of what is known as Hilbert problem. As the problem and the solution is not that well known, it is described briefly. For details one might see [6].

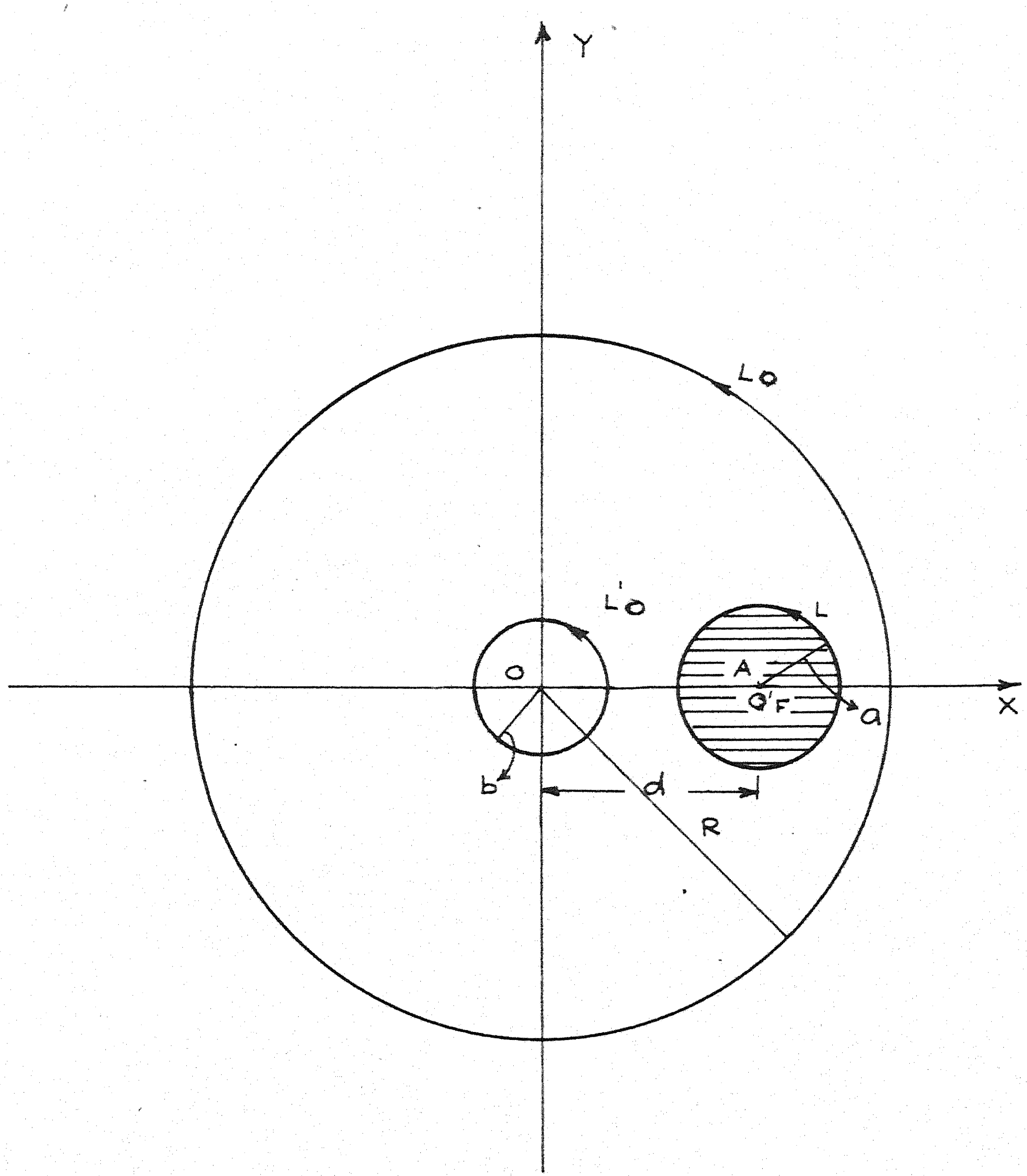


FIG.3

Let there be a line L. The line may be the union of either smooth arcs or contours or both. This line L divides the region R under consideration into two parts. That part of the region R which lies towards the left of L (looking in the anticlockwise direction) is named R_1 and the other as R_s. Let t be any point on L, then the boundary values of F(z), as z approaches t from the left is denoted by $F^+(t)$ and the other by $F^-(t)$. Now let G(t) and L(t) be two functions which under suitable conditions [6], satisfy the equation

$$F^+(t) = G(t)\ F^-(t) + L(t) \quad \text{on } L . \tag{4.1}$$

Hilbert problem is to find the sectionally holomorphic function F(z), which satisfies the above condition (4.1). If L(t) = 0, the problem is called homogeneous.

We shall not need the solution of the problem for a general value of G(t). We shall deal with the case when G(t) = 1. Hence the problem (4.1) reduces to the determination of a sectionally holomorphic function F(z) such that

$$F^+(t) - F^-(t) = L(t) \quad \text{on } L .$$

One might call L(t) to be the discontinuity of F(z) on L. Using the well known theorems of Plemelj, J. [6], a particular solution of the problem is atonce written as

$$F_0(z) = \frac{1}{2\pi i} \int_L \frac{L(t)}{t - z}\, dt.$$

The solution $F_o(z)$ is a sectionally holomorphic function which vanishes at infinity and for which

$$F_o^+(t) - F_o^-(t) = L(t) \quad \text{on } L .$$

Next consider the difference $F(z) - F_o(z) = F_*(z)$ where $F_*(z)$ is an unknown solution. It follows that

$$F_*^+(t) - F_*^-(t) = 0 \quad \text{on } L.$$

Thus on the basic of a known property of functions of a complex variable [12], the values of $F_*(z)$ on the left and the right of L continue each other analytically. Therefore if one prescribes for the function $F_*(z)$ suitable values on L, this function will be holomorphic in the entire plane. Consequently, by the Liouville theorem $F_*(z)$ = constant in the entire plane and the general solution of the Hilbert problem is given by $F(z) = F_o(z) + C$ or

$$F(z) = \frac{1}{2\pi i} \int_L \frac{L(t)}{t - z}\, dt + C$$

where C is any arbitrary constant. If it is required that $F(\infty) = 0$, then one has to take $C = 0$. Further, if $F(z)$ is a sectionally holomorphic function in some region which does not coincide with the whole plane, then this function may always be represented in the form of the sum of a function holomorphic in R and a Cauchy integral.

$$F(z) = \frac{1}{2\pi i} \int_L \frac{L(t)}{t - z}\, dt + F^*(z) \qquad (4.2)$$

where L is the line of discontinuity and $L(t) = F^+(t) - F^-(t)$ on L and $F^*(z)$ is a function holomorphic in R. These results are enough to derive the solution of the problem stated at the begining of this chapter. We now restate the problem in more explicit terms.

Consider a circular disc of radius R with two holes, one concentric and the other eccentric, of radii b and a respectively. The two holes do not touch or intersect each other. Let the boundary of the concentric hole be L_o' and that of the eccentric hole be L and let the outer boundary be L_o. The distance between the centres of the two holes is d. Note that $b < |d| < R - a$. As stated above, the centre is taken as the origin. The X-axis is taken as the line passing through the centre and the centre of the eccentric hole and the Y-axis is taken perpendicular to it lying in the plane. The disc with the two holes will be referred hereafter as shell.

In the eccentric hole of the shell, a circular disc of radius a is embedded. In the absence of shell, the insert has a spontaneous homogeneous deformation prescribed by $u = \delta(x - d)$, $v = \delta y$ where δ lies within elastic limits. The common boundary of the insert and the shell is welded to avoid slipping. The composite disc with the hole at the centre, is rotating about an axis through the origin perpendicular to the plane, with a constant angular velocity, say ω. Let the region occupied by the insert be denoted by R_1 and the one occupied by the shell by R_s.

As noted in the earlier chapters, the whole problem depends on finding two sets of analytic functions $\{\phi_i(z),\ \psi_i(z)\}$ and $\{\phi_s(z),\ \psi_s(z)\}$ in the insert and the shell respectively which satisfy all the relevant equations of elasticity theory and the prescribed boundary conditions. The boundary conditions are 1) the boundaries L_o and L_o' are free from tractions 2) the radial and shearing stresses are continuous at the boundary L 3) the discontinuity between the displacement components of the shell and the insert are prescribed.

Let (u_i, v_i) denote the displacement components in the insert and (u_s, v_s) in the shell. For the sake of generality, the discontinuity for a passage through L, of the holomorphic function may be defined by

$$u_s - u_i = g_1(t), \quad v_s - v_i = g_2(t)$$

where $t = x + iy$ and $g_1(t)$, $g_2(t)$ are known functions of t. If f(t) is defined as in (1.20) and the boundary tractions are prescribed at L_o, it is seen that

$$\phi(t) + t\,\overline{\phi'(t)} + \overline{\psi(t)} = f(t) \text{ on } L_o . \tag{4.3}$$

Also by the continuity of P_{rr}, $P_{r\theta}$, we get

$$\phi_i(t) + t\,\overline{\phi_i'(t)} + \overline{\psi_i(t)} = \phi_s(t) + t\,\overline{\phi_s'(t)} + \overline{\psi_s(t)} \text{ on L.} \tag{4.4}$$

Since the discontinuity in the displacement components are prescribed on L

$$k\,\phi_1(t) - t\,\overline{\phi_1'(t)} - \overline{\psi_1(t)} = k\,\phi_s(t) - t\,\overline{\phi_s'(t)} - \overline{\psi_s(t)}$$

$$+ 2\mu\left\{g_1(t) + i\,g_2(t)\right\} \text{ on } L. \qquad (4.5)$$

It follows from (4.4), (4.5) that

$$\phi_1(t) - \phi_s(t) = \frac{2\mu}{k+1}\left\{g_1(t) + i\,g_2(t)\right\} \text{ on } L. \qquad (4.6)$$

Further taking the conjugate complex form (4.4) and using (4.6), one obtains

$$\psi_1(t) - \psi_s(t) = -\frac{2\mu}{k+1}\left\{\overline{g_1(t)} + i\,\overline{g_2(t)} + \bar{t}(g_1'(t) + i\,g_2'(t))\right\} \text{ on } L. \qquad (4.7)$$

It follows from (4.6) and (4.7) that

$$\phi(z) = \phi_o(z) + \frac{\mu}{\pi i(k+1)} \int_L \frac{g_1(t) + i\,g_2(t)}{t - z}\,dt,$$

$$\psi(z) = \psi_o(z) + \frac{\mu}{\pi i(k+1)} \int_L - \frac{\overline{g_1(t)} + i\,\overline{g_2(t)} + \bar{t}\quad g_1'(t) + i\,g_2'(t)}{t - z}\,dt \qquad (4.8)$$

where $\left\{\phi_o(z),\ \psi_o(z)\right\}$ are functions holomorphic in the region R ($= R_1 \cup R_s$). For the sake of brevity, let us write

$$\phi_*(z) = \frac{\mu}{\pi i(k+1)} \int_L \frac{g_1(t) + i\,g_2(t)}{t - z}\,dt,$$

$$\psi_*(z) = \frac{\mu}{\pi i(k+1)} \int_L - \frac{\overline{g_1(t)} + i\,\overline{g_2(t)} + \bar{t}(g_1'(t) + i\,g_2'(t))}{t - z}\,dt. \qquad (4.9)$$

The equations (4.8) become

$$\phi(z) = \phi_o(z) + \phi_*(z) \ , \quad \psi(z) = \psi_o(z) + \psi_*(z) .$$

The functions $\{\phi_o(z), \psi_o(z)\}$ are yet to be found. But $\phi_*(z)$ and $\psi_*(z)$ are known functions defined by (4.9). Substituting (4.8) into (4.3), one obtains

$$\phi_o(t) + t\,\overline{\phi_o'(t)} + \overline{\psi_o(t)} = f_o(t) \quad \text{on } L_o$$

where $f_o(t) = f^{(2)}(t) - t\,\overline{\phi_*'(t)} - \overline{\psi_*(t)} - \phi_*(t)$ is a function known on L_o. One has thus arrived at the usual first fundamental boundary value problem for the body which presents no problem for its solution. After having determined $\{\phi_o(z), \psi_o(z)\}$, the functions $\{\phi(z), \psi(z)\}$ are calculated with the help of (4.8) and these complex potentials give the stress field everywhere. In the present problem

$$g_1(t) = -\delta\left(\frac{t+\bar{t}}{2} - d\right), \quad g_2(t) = -\delta\left(\frac{t+\bar{t}}{2i}\right) .$$

Then from (4.9)

$$\begin{aligned}
\phi_*(z) &= -\frac{2\mu\delta}{1+k}(z-d) && \text{for } z \text{ in } R_1 , \\
&= 0 && \text{for } z \text{ in } R_s , \\
\psi_*(z) &= \frac{2\mu\delta d}{1+k} && \text{for } z \text{ in } R_1 , \\
&= -\frac{4\mu\delta}{k+1}\,\frac{a^2}{(z-d)} && \text{for } z \text{ in } R_s .
\end{aligned} \tag{4.10}$$

The boundary condition on L_o is

$$\phi_o(t) + t\,\overline{\phi_o'(t)} + \overline{\psi_o(t)} = f_o(t)$$

where

$$f_o(t) = f^{(2)}(t) - \phi_*(t) - t\,\overline{\phi_*'(t)} - \overline{\psi_*(t)}$$

and

$$f^{(2)}(t) = i \int^{s} \left\{ (P_{nx} - P_{nx}^{(1)}) + i(P_{ny} - P_{ny}^{(1)}) \right\} ds .$$

Here $P_{n\alpha}$ $(\alpha = x,y)$ is the given stress vector on L_o and $P_{n\alpha}^{(1)}$ $(\alpha = x,y)$ are given by (1.18). The boundary condition on L_o reduces to

$$\phi_o(t) + t\,\overline{\phi_o'(t)} + \overline{\psi_o(t)} = \frac{3+\nu}{8}\,\rho\,\omega^2 R^2 t + \frac{4\mu\delta}{k+1}\,\frac{a^2}{\bar{t} - d} . \quad (4.11)$$

Similarly on L_o', the boundary condition reduces to

$$\phi_o(t) + t\,\overline{\phi_o'(t)} + \overline{\psi_o(t)} = -\frac{3+\nu}{8}\,\rho\,\omega^2 b^2 t + \frac{4\mu\delta}{k+1}\,\frac{a^2}{\bar{t} - d} + c_1 .$$

(4.12)

The functions $\{\phi_o(z),\, \psi_o(z)\}$ are holomorphic in the annular ring between L_o' and L_o and hence these functions may be expanded in series.

$$\phi_o(z) = \sum_{n=-\infty}^{\infty} a_n z^n , \quad \psi_o(z) = \sum_{n=-\infty}^{\infty} b_n z^n . \quad (4.13)$$

Substituting the forms of $\{\phi_o(z),\, \psi_o(z)\}$ from (4.13) in (4.11) and putting $t = Re^{i\theta}$ and $be^{i\theta}$ respectively and equating the like powers

of θ, the following equations are obtained.

$$a_o + 2\,\bar{a}_2 R^2 + b_o = 0,$$

$$(a_1 + \bar{a}_1) R + \frac{\bar{b}_{-1}}{R^2} = \frac{3+\nu}{8} \rho \omega^2 R^3 + \frac{4\mu\delta}{k+1} \frac{a^2}{R},$$

$$a_2 R^2 + \frac{\bar{b}_{-2}}{R^2} = \frac{4\mu\delta}{k+1} \frac{a^2 d}{R^2},$$

$$a_{n+2} R^{n+2} - \frac{n\,\bar{a}_{-n}}{R^n} + \frac{\bar{b}_{-(n+2)}}{R^{n+2}} = \frac{4\mu\delta}{k+1} \frac{a^2 d^{n+1}}{R^{n+2}}, \quad (n \geq 1),$$

$$\frac{a_{-n}}{R^n} + (n+2)\,\bar{a}_{n+2} R^{n+2} + \bar{b}_n R^n = 0, \quad (n \geq 1).$$

(4.14)

Similarly from (4.12) the following equations are obtained.

$$a_o + 2\,\bar{a}_2 b^2 + \bar{b}_o = -\frac{4\mu\delta}{k+1} \frac{a^2}{d} + c_1,$$

$$(a_1 + \bar{a}_1)\, b + \frac{\bar{b}_{-1}}{b} = -\frac{(3+\nu)}{8} \rho \omega^2 b^3,$$

$$a_2 b^2 + \frac{\bar{b}_{-2}}{b^2} = 0,$$

$$a_{n+2} b^{n+2} - \frac{n\,\bar{a}_{-n}}{b^n} + \frac{\bar{b}_{-(n+2)}}{b^{n+2}} = 0, \quad (n \geq 1),$$

$$\frac{a_{-n}}{b^n} + (n+2)\,\bar{a}_{n+2} b^{n+2} + \bar{b}_n b^n = -\frac{4\mu\delta}{k+1} \frac{a^2 b^n}{d^{n+1}}, \quad (n \geq 1).$$

(4.15)

Solving these equations (4.14) and (4.15), the unknown constants are obtained as follows.

$$a_o + b_o = -\frac{4\mu\delta}{k+1}\ \frac{2da^2R^2}{R^4 - b^4} \quad ,$$

$$c_1 = \frac{4\mu\delta a^2}{k+1}\left\{-\frac{2d}{R^2+b^2} + \frac{1}{d}\right\} \quad ,$$

$$a_1 = \frac{3+\nu}{16}\,\rho\,\omega^2\ \frac{(R^4+b^4)}{R^2-b^2} + \frac{2\mu\delta}{k+1}\quad\frac{a^2}{R^2-b^2} \quad ,$$

$$a_2 = \frac{4\mu\delta}{k+1}\ \frac{a^2d}{R^4-b^4} \quad ,$$

$$b_{-1} = -\frac{(3+\nu)}{8}\,\rho\,\omega^2\ \frac{R^2b^2(R^2+b^2)}{R^2-b^2} - \frac{4\mu\delta}{k+1}\ \frac{a^2b^2}{R^2-b^2} \quad ,$$

$$b_{-2} = -\frac{4\mu\delta}{k+1}\ \frac{a^2b^4d}{R^4-b^4} \quad ,$$

$$a_n = \frac{4\mu\delta a^2}{k+1}\ \frac{R^{2n}b^{2n}\left\{(n+2)\,d^{n+1}(R^2-b^2) - \dfrac{R^{2n+4}-b^{2n+4}}{d^{n+1}}\right\}}{\left\{(R^{2n+4}-b^{2n+4})(R^{2n}-b^{2n}) - n(n+2)\,R^{2n}b^{2n}(R^2-b^2)^2\right\}},$$

$$(\, n \geq 1 \,) \quad ,$$

$$b_n = -\frac{4\mu\delta a^2}{k+1}\;\frac{\left[(n+2)\,d^{n+1}\left(R^{2n+2}-b^{2n+2}\right)-\dfrac{b^{2n}}{d^{n+1}}\left\{\left(R^{2n+4}-b^{2n+4}\right)+n(n+2)\,R^{2n+2}\left(R^2-b^2\right)\right\}\right]}{\left\{\left(R^{2n+4}-b^{2n+4}\right)\left(R^{2n}-b^{2n}\right)-n(n+2)\,R^{2n}b^{2n}\left(R^2-b^2\right)^2\right\}},$$

$$(n \geq 1),$$

$$a_{n+2} = \frac{4\mu\delta a^2}{k+1}\;\frac{\left[d^{n+1}\left(R^{2n}-b^{2n}\right)-\dfrac{nR^{2n}b^{2n}\left(R^2-b^2\right)}{d^{n+1}}\right]}{\left\{\left(R^{2n+4}-b^{2n+4}\right)\left(R^{2n}-b^{2n}\right)-n(n+2)\,R^{2n}b^{2n}\left(R^2-b^2\right)^2\right\}},$$

$$(n \geq 1),$$

$$b_{-(n+2)} = -\frac{4\mu\delta a^2}{k+1}\;\frac{\left[\dfrac{nR^{2n+2}b^{2n+2}\left(R^{2n+2}-b^{2n+2}\right)}{d^{n+1}}-d^{n+1}b^{2n+2}\times\left\{n(n+2)\,R^{2n+2}-(n+1)^2R^{2n}b^2+b^{2n+2}\right\}\right]}{\left\{\left(R^{2n+4}-b^{2n+4}\right)\left(R^{2n}-b^{2n}\right)-n(n+2)\,R^{2n}b^{2n}\left(R^2-b^2\right)^2\right\}},$$

$$(n \geq 1).$$

(4.16)

Substitution of the values of the unknowns from (4.16) in $\phi_0(z)$ and $\psi_0(z)$ determine them explicitly apart from a constant, which will not affect the stress field. The stress field in the insert for the homogeneous part of the equations is given by $\{\phi_1(z), \psi_1(z)\}$ where

$$\phi_1(z) = \phi_0(z) - \frac{2\mu\delta}{k+1}(z - d) \ ,$$

$$\psi_1(z) = \psi_0(z) + \frac{2\mu\delta d}{k+1}$$

and for the shell

$$\phi_s(z) = \phi_0(z) \ ,$$

$$\psi_s(z) = \psi_0(z) - \frac{4\mu\delta}{k+1} \ \frac{a^2}{(z - d)} \ .$$

The stress field in the non-homogeneous case may be obtained by adding the particular integrals given by (1.15) to the stresses obtained in the homogeneous problem.

The particular case of a rotating disc with an eccentric circular insert can be deduced by putting $b = 0$. In that case the constants turn out to be

$$a_0 = 0 \ ,$$

$$b_0 = -\frac{8\mu\delta}{k+1} \ \frac{a^2 d}{R^2} \ ,$$

$$a_1 = \frac{3+\nu}{16}\rho\omega^2 R^2 + \frac{2\mu\delta}{k+1} \ \frac{a^2}{R^2} \ ,$$

$$a_2 = \frac{4\mu\delta}{k+1} \ \frac{a^2 d}{R^4} \ ,$$

$$a_{-n} = b_{-(n+2)} = 0 \ , \quad (n \geq 1) \ ,$$

$$a_{n+2} = \frac{4\mu\delta}{k+1} \ \frac{a^2 d^{n+1}}{R^{2n+4}} \ , \quad (n \geq 1) \ ,$$

$$b_n = - \frac{4\mu\delta}{k+1} \ \frac{(n+2)\, a^2 d^{n+1}}{R^{2n+2}} \ , \ (n \geq 1) \ .$$

It may be seen that the complex potentials for the homogeneous part of the equations in the case of insert are given by

$$\phi_1(z) = \frac{3+\nu}{16} \rho \omega^2 R^2 z + \frac{2\mu\delta}{k+1} \frac{(R^2 + dz)}{R^2 - dz} \ \frac{a^2 z}{R^2} - \frac{2\mu\delta}{k+1} (z - d) \ ,$$

$$\psi_1(z) = \frac{2\mu\delta d}{k+1} - \frac{4\mu\delta}{k+1} \ \frac{a^2 d(2R^2 - dz)}{(R^2 - dz)^2}$$

and for the shell

$$\phi_s(z) = \frac{3+\nu}{16} \rho \omega^2 R^2 z + \frac{2\mu\delta}{k+1} \ \frac{(R^2 + dz)}{(R^2 - dz)} \ \frac{a^2 z}{R^2} \ ,$$

$$\psi_s(z) = - \frac{4\mu\delta}{k+1} \left\{ \frac{a^2 d(2R^2 - dz)}{(R^2 - dz)^2} + \frac{a^2}{(z - d)} \right\} . \tag{4.17}$$

This result, when $\omega = 0$ tallies with those obtained by Gupta, S. C. [13] .

It may again be seen that putting $b = 0$ and $d = 0$, one obtains the case of a rotating composite disc containing a

concentric circular inclusion. In this case the actual stresses are the following.

For the insert

$$P_{rr} = \frac{\rho\omega^2}{8}(3+\nu)(R^2 - r^2) + \frac{4\mu\delta}{k+1}\ \frac{(a^2 - R^2)}{R^2} ,$$

$$P_{\theta\theta} = \frac{\rho\omega^2}{8}\ \left\{R^2(3+\nu) - r^2(1+3\nu)\right\} + \frac{4\mu\delta}{k+1}\ \frac{(a^2 - R^2)}{R^2} ,$$

$$P_{r\theta} = 0$$

and for the shell

$$P_{rr} = \frac{\rho\omega^2}{8}(3+\nu)(R^2 - r^2) + \frac{4\mu\delta}{k+1}\ \frac{a^2}{R^2}(1 - \frac{R^2}{r^2}) ,$$

$$P_{\theta\theta} = \frac{\rho\omega^2}{8}\left\{R^2(3+\nu) - r^2(1+3\nu)\right\} + \frac{4\mu\delta}{k+1}\ \frac{a^2}{R^2}(1+ \frac{R^2}{r^2}) ,$$

$$P_{r\theta} = 0 .$$

These results tally with those obtained in the second chapter (2.18).

Numerical work for the calculation of stresses both in the insert and in the shell has been done on Computer CDC 3600. The Poisson's ratio was taken as 1/3 and δ =0.01. The value of $\rho\,\omega^2/(\mu/k+1)$ was taken to be 0.005. The outer radius of the disc was taken to be 1.5 and the radius of the concentric hole is 0.1. The radius of the eccentric hole is 0.3. The distance between the

centres is d. Four cases have been discussed, with $d = 0.7, 0.8, 0.9, 1.0$ respectively. The normal and shearing stresses and the hoop stress for the insert and the shell at equilibrium boundary were found and are given in the tables.

The first table 4.1 gives the values of some of the coefficients of z in the power series of ϕ and ψ. Note that the coefficients decrease very fast. In the particular case the number, n, of terms in the power series was taken as 20. In the table 4.1, it may be observed that all coefficients corrected to eight decimal places are zero after thirteen terms. The table 4.2 is for $d = 0.7$. It may be recalled that d is the distance between the centres of the shell and the insert. The first column in table 4.2 gives the angular distance θ in degrees, with the origin at the centre of the insert, measured from the X-axis in anti-clockwise direction. The second and third columns give the normal and shearing stresses, $P_{rr}(k+1)/\mu$, $P_{r\theta}(k+1)/\mu$ respectively at the equilibrium boundary. These are the same(as they should be) for the insert and the shell. The 4th and 5th columns give the hoop stress for the shell and the insert at the equilibrium boundary. The last column gives the hoop stress on the inner boundary of the ring for $b = 0.1$. In the tables 4.3, 4.4, 4.5, the same results are repeated for $d = 0.8, 0.9, 1.0$ respectively.

TABLE 4.1

n	a_n	b_n	a_{-n}	b_{-n}
1	0.00315783	-0.00093159	-0.00006420	-0.00006337
2	0.00049779	-0.00043184	-0.00000101	-0.00000005
3	0.00014224	-0.00016861	-0.00000002	-0.00000064
4	0.00004800	-0.00006296	0.00000000	-0.00000002
5	0.00001499	-0.00002285	0.00000000	-0.00000000
6	0.00000466	-0.00000812	0.00000000	0.00000000
7	0.00000145	-0.00000284	0.00000000	0.00000000
8	0.00000045	-0.00000098	0.00000000	0.00000000
9	0.00000014	-0.00000034	0.00000000	0.00000000
10	0.00000004	-0.00000011	0.00000000	0.00000000
11	0.00000001	-0.00000004	0.00000000	0.00000000
12	0.00000000	-0.00000001	0.00000000	0.00000000
13	0.00000000	-0.00000000	0.00000000	0.00000000
14	0.00000000	0.00000000	0.00000000	0.00000000
15	0.00000000	0.00000000	0.00000000	0.00000000
16	0.00000000	0.00000000	0.00000000	0.00000000
17	0.00000000	0.00000000	0.00000000	0.00000000
18	0.00000000	0.00000000	0.00000000	0.00000000
19	0.00000000	0.00000000	0.00000000	0.00000000
20	0.00000000	0.00000000	0.00000000	0.00000000

TABLE 4.2

$d = 0.70$

θ	$\frac{P_{rr}(k+1)}{\mu}$	$\frac{P_{r\theta}(k+1)}{\mu}$	$\frac{P_{\theta\theta}(k+1)}{\mu}$ Shell	$\frac{P_{\theta\theta}(k+1)}{\mu}$ Insert	$\frac{P_{\theta\theta}(k+1)}{\mu}$ On the concentric hole
0	-0.03232	0.00000	0.04881	-0.03119	0.04898
30	-0.03258	0.00022	0.04864	-0.03136	0.02458
60	-0.03288	0.00052	0.04806	-0.03194	-0.00872
90	-0.03272	0.00073	0.04729	-0.03271	-0.01934
120	-0.03245	0.00046	0.04692	-0.03308	0.00664
150	-0.03224	0.00049	0.04723	-0.03277	0.02449
180	-0.03097	0.00000	0.04673	-0.03327	0.03137

TABLE 4.3

$d = 0.80$

θ	$\frac{P_{rr}(k+1)}{\mu}$	$\frac{P_{r\theta}(k+1)}{\mu}$	$\frac{P_{\theta\theta}(k+1)}{\mu}$ Shell	$\frac{P_{\theta\theta}(k+1)}{\mu}$ Insert	$\frac{P_{\theta\theta}(k+1)}{\mu}$ On the concentric hole
0	-0.03104	0.00000	0.05052	-0.02948	0.03683
30	-0.03163	0.00024	0.05020	-0.02980	0.02104
60	-0.03238	0.00082	0.04922	-0.03079	-0.00134
90	-0.03231	0.00121	0.04793	-0.03207	-0.00376
120	-0.03186	0.00093	0.04706	-0.03294	0.00826
150	-0.03149	0.00062	0.04681	-0.03319	0.02056
180	-0.03099	0.00000	0.04657	-0.03343	0.02534

TABLE 4.4

$d = 0.90$

θ	$\frac{P_{rr}(k+1)}{\mu}$	$\frac{P_{r\theta}(k+1)}{\mu}$	$\frac{P_{\theta\theta}(k+1)}{\mu}$ Shell	$\frac{P_{\theta\theta}(k+1)}{\mu}$ Insert	$\frac{P_{\theta\theta}(k+1)}{\mu}$ On the concentric hole
0	-0.02867	0.00000	0.05375	-0.02625	0.02821
30	-0.02986	0.00038	0.05297	-0.02703	0.01825
60	-0.03140	0.00133	0.05103	-0.02897	0.00387
90	-0.03150	0.00188	0.04885	-0.03115	0.00211
120	-0.03087	0.00153	0.04734	-0.03266	0.00970
150	-0.03038	0.00082	0.04672	-0.03328	0.01753
180	-0.03013	0.00000	0.04654	-0.03346	0.02055

TABLE 4.4

d = 0.90

θ	$\frac{P_{rr}(k+1)}{\mu}$	$\frac{P_{r\theta}(k+1)}{\mu}$	$\frac{P_{\theta\theta}(k+1)}{\mu}$ Shell	$\frac{P_{\theta\theta}(k+1)}{\mu}$ Insert	$\frac{P_{\theta\theta}(k+1)}{\mu}$ On the concentric hole
0	-0.02867	0.00000	0.05375	-0.02625	0.02821
30	-0.02986	0.00038	0.05297	-0.02703	0.01825
60	-0.03140	0.00133	0.05103	-0.02897	0.00387
90	-0.03150	0.00188	0.04885	-0.03115	0.00211
120	-0.03087	0.00153	0.04734	-0.03266	0.00970
150	-0.03038	0.00082	0.04672	-0.03328	0.01753
180	-0.03013	0.00000	0.04654	-0.03346	0.02055

TABLE 4.5

$d = 1.0$

θ	$\frac{P_{rr}(k+1)}{\mu}$	$\frac{P_{r\theta}(k+1)}{\mu}$	$\frac{P_{\theta\theta}(k+1)}{\mu}$ Shell	$\frac{P_{\theta\theta}(k+1)}{\mu}$ Insert	$\frac{P_{\theta\theta}(k+1)}{\mu}$ On the concentric hole
0	-0.02415	0.00000	0.06062	-0.01938	0.02154
30	-0.02655	0.00102	0.05826	-0.02174	0.01593
60	-0.02958	0.00233	0.05385	-0.02615	0.00788
90	-0.02999	0.00292	0.05007	-0.02993	0.00693
120	-0.02918	0.00236	0.04774	-0.03226	0.01101
150	-0.02857	0.00118	0.04678	-0.03322	0.01504
180	-0.02839	0.00000	0.04658	-0.03342	0.01655

CHAPTER 5

Rotating Disc With Two Eccentric Circular Inserts

The problem of a rotating disc with two eccentric circular inserts is considered. It may be emphasised at this stage, that we have considered rather a general case, inasmuchas, the inserts may be of different radii and at different distances from the centre of the disc. But the line of centres is passing through the centre of the shell. It may be remarked that, even the case when the centres of the inserts and the shell are not in the same straight line, can also be done by applying the same technique, but the algebra will be a little heavier. The inserts are supposed to undergo a prescribed spontaneous homogeneous deformations in the absence of the shell. The composite disc is rotating with a constant angular velocity, about an axis through the centre of the shell perpendicular to its plane. Because of the misfits in size, and because of rotation, stresses develop both in the inserts and the shell. The outer boundary is free from external tractions. The elastic field is calculated everywhere by finding three sets of complex potential functions: two for the inserts and one for the shell. These functions suffer a discontinuity at the interface between the shell and the insert. The method of solution for the present problem is based upon Hilbert's theorem which states that a sectionally holomorphic function can be represented in the form

of the sum of a holomorphic function and a Cauchy integral, evaluated along the line of discontinuity, provided the region is finite.

Consider a circular disc of radius R with two eccentric holes of radii a_1 and a_2 situated at distances d_1 and d_2 from the centre O, which is taken as the origin (Fig. 4, p.71). All the three centres are taken to be collinear and the line joining them is taken as the X-axis and a line perpendicular to it at O, in the plane is taken as the Y-axis. Note that $|d_1| < R - a_1$ and $|d_2| < R - a_2$. Also the two inserts do not touch or overlap each other. Let the outer boundary of the disc be L_o.

We imagine that there are two holes of radii a_1 and a_2 in the shell. We call their boundaries as L_1 and L_2. In these holes, materials of slightly larger radii (within elastic limits) are inserted in the manner described in chapter 3 and welded at the rim to avoid slipping. In the absence of the outer material, the spontaneous deformation is characterised by $u = \delta_1(x-d_1)$, $v = \delta_1 y$ and $u = \delta_2(x-d_2)$, $v = \delta_2 y$ for the two inserts. The constants δ_1 and δ_2 lie within the elastic limits. The composite disc is rotating about the axis through the centre perpendicular to the plane with a constant angular velocity, say ω. Let the regions occupied by the two inserts be R_{11} and R_{12} and that by the shell be R_s. To find the elastic field, we have to find three sets of complex potentials : $\left\{\phi_{11}(z), \psi_{11}(z)\right\}$, $\left\{\phi_{12}(z), \psi_{12}(z)\right\}$ and

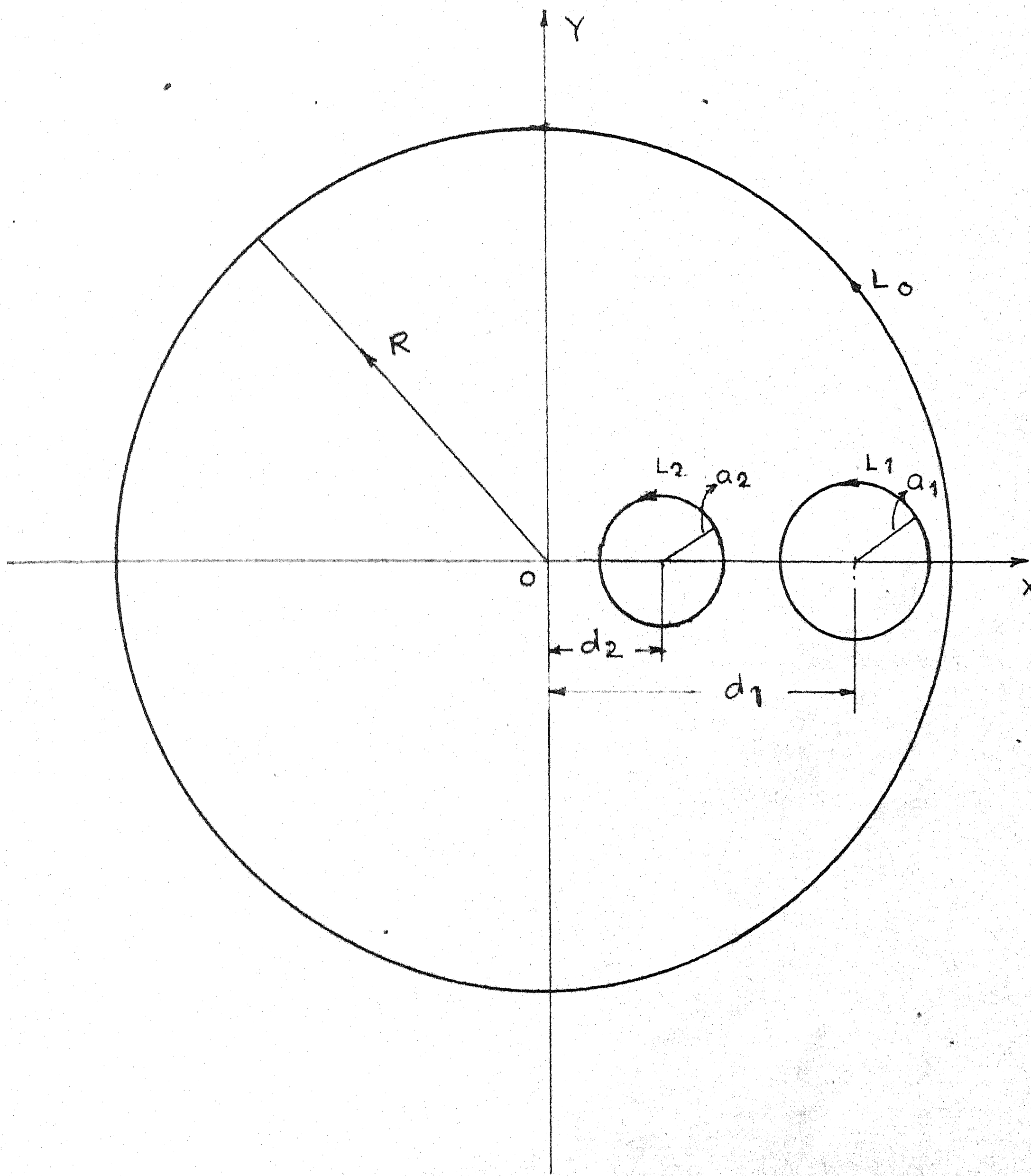


FIG. 4

$\{\phi_s(z), \ \psi_s(z)\}$, the first two for the inserts and the last for the shell. In this case, the line of discontinuity L is the union of the contours L_1 and L_2.

As noted in the previous chapter, if $F(z)$ is the sectionally holomorphic function to be found in the region R and if the discontinuity of $F(z)$ be $L(t)$ along the interface i.e., if

$$F^+(t) - F^-(t) = L(t) \quad \text{on } L ,$$

then

$$F(z) = F^*(z) + \frac{1}{2\pi i} \int_L \frac{L(t)}{t - z} \, dt$$

where t is any point on L and $F^*(z)$ is the function analytic in the whole region. Other symbols have the meanings given in chapter 4. $F^*(z)$ is evaluated by the boundary conditions.

Let (u_i , v_i) and (u_s , v_s) be the displacement components of the inserts and the shell respectively. At the equilibrium interface L_1 of the regions R_{i1} and R_s

$$u_{i1}^b - u_s^b = - \delta_1(x - d_1) \ , \quad v_{i1} - v_s = - \delta_1 y \ .$$

The superscript b denotes the value of the displacement component at the boundary. It may be emphasized that (u_{i1}^b, v_{i1}^b) - the displacement at the boundary of the insert - is measured from its natural state i.e., the state it would have in the absence of the shell, while (u_s^b , v_s^b) is the displacement at the boundary of the shell measured from its natural state, which is the state when

the inner hole is of radius a_1.

As stated in chapter 4, $\phi(z) = \phi_0(z) + \phi_*(z)$ and $\psi(z) = \psi_0(z) + \psi_*(z)$ where $\{\phi_0(z), \psi_0(z)\}$ are functions analytic in the whole region R $(= R_{11} \cup R_{12} \cup R_s)$ and $\{\phi_*(z), \psi_*(z)\}$ are defined as follows.

$$\phi_*(z) = \frac{\mu}{k+1} \frac{1}{\pi i} \int_L \frac{g_1(t) + i\, g_2(t)}{t - z} dt ,$$

$$\psi_*(z) = \frac{\mu}{k+1} \frac{1}{\pi i} \int_L - \frac{\overline{g_1(t)} + i\, \overline{g_2(t)} + \bar{t}(g_1'(t) + i\, g_2'(t))}{t - z} dt .$$

Note that, in the present case

$$g_1(t) + i\, g_2(t) = - \delta_1(t - d_1) \quad \text{on } L_1 ,$$

and

$$- (\overline{g_1(t)} + i\, \overline{g_2(t)} + \bar{t}\, g_1'(t) + i\, \bar{t}\, g_2'(t)) = 2\delta_1 \bar{t} - \delta_1 d_1 \quad \text{on } L_1 ,$$

$$= 2\delta_2 \bar{t} - \delta_2 d_2 \quad \text{on } L_2 .$$

Also,

$$\bar{t} = d_1 + \frac{a_1^2}{t - d_1} \quad \text{on } L_1 ,$$

$$= d_2 + \frac{a_2^2}{t - d_2} \quad \text{on } L_2 .$$

It may be easily seen that

$$\phi_*(z) = -\frac{2\mu\delta_1}{k+1}(z - d_1) \qquad \text{for } z \text{ in } R_{11} ,$$

$$= -\frac{2\mu\delta_2}{k+1}(z - d_2) \qquad \text{for } z \text{ in } R_{12} ,$$

$$= 0 \qquad \text{for } z \text{ in } R_s . \tag{5.1}$$

$$\psi_*(z) = \frac{2\mu}{k+1}\left\{\delta_1 d_1 - \frac{2\delta_2 a_2^2}{z - d_2}\right\} \qquad \text{for } z \text{ in } R_{11} ,$$

$$= \frac{2\mu}{k+1}\left\{\delta_2 d_2 - \frac{2\delta_1 a_1^2}{z - d_1}\right\} \qquad \text{for } z \text{ in } R_{12} ,$$

$$= -\frac{4\mu}{k+1}\left\{\frac{\delta_1 a_1^2}{z - d_1} + \frac{\delta_2 a_2^2}{z - d_2}\right\} \qquad \text{for } z \text{ in } R_s . \tag{5.2}$$

The boundary condition that is to be satisfied on the outer boundary L_o is

$$\phi_o(t) + t\,\overline{\phi_o'(t)} + \overline{\psi_o(t)} = f_o(t) \tag{5.3}$$

where

$$f_o(t) = f^{(2)}(t) - \phi_*(t) - t\,\overline{\phi_*'(t)} - \overline{\psi_*(t)} .$$

Here

$$f^{(2)}(t) = i\int^{s}\left\{(P_{nx} - P_{nx}^{(1)}) + i\,(P_{ny} - P_{ny}^{(1)})\right\}ds$$

where $P_{n\alpha}$ $(\alpha = x,y)$ is the given stress vector on the boundary L_o

and $P^{(1)}_{n\alpha}$ $(\alpha = x,y)$ given by (1.18), is the induced stress vector on the boundary because of the rotation of the body. In the present case $P_{n\alpha}$ $(\alpha = x,y)$ is zero since the outer boundary is free from tractions and so

$$f^{(1)}(t) = \frac{3+\nu}{8}\,\rho\,\omega^2 R^2 t \quad \text{on } L_0 \,.$$

The boundary condition (5.3) on L_0 reduces to

$$\phi_0(t) + t\,\overline{\phi_0'(t)} + \overline{\psi_0(t)} = \frac{3+\nu}{8}\,\rho\,\omega^2 R^2 t + \frac{4\mu}{k+1}\left\{\frac{\delta_1 a_1^2}{\bar{t}-d_1} + \frac{\delta_2 a_2^2}{\bar{t}-d_2}\right\} \tag{5.4}$$

where t is a point on L_0. The region R is mapped on the unit circle $|\xi|<1$ by the transformation $z = R\,\xi$. Then the above boundary condition (5.4) takes the form

$$\phi_{10}(\sigma) + \sigma\,\overline{\phi_{10}'(\sigma)} + \overline{\psi_{10}(\sigma)} = \frac{3+\nu}{8}\,\rho\,\omega^2 R^3 \sigma + \frac{4\mu}{k+1}\left\{\frac{\delta_1 a_1^2}{R\bar{\sigma}-d_1} + \frac{\delta_2 a_2^2}{R\bar{\sigma}-d_2}\right\} \tag{5.5}$$

where $\bar{\sigma}$ is a point on the unit circle. The values of the functions $\phi_{10}(\xi)$ and $\psi_{10}(\xi)$ are to be found from the above equation (5.5) using the integro-differential technique, which is described below very briefly. For details one might see [6, 8].

This method essentially starts with mapping the connected region (which may be finite or infinite) to the unit circle in the

ξ - plane by a mapping function of the type $z = W(\xi)$ and to obtain the boundary condition of the type

$$\alpha\,\phi(\sigma) + \frac{W(\sigma)}{\overline{W'(\sigma)}}\,\overline{\phi'(\sigma)} + \overline{\psi(\sigma)} = H(\sigma)\ . \tag{5.6}$$

Note that σ is the boundary point on the circle. We now multiply by $\frac{1}{2\pi i}\,\frac{d\sigma}{\sigma - \xi}$ $(\,|\xi| < 1\,)$ and integrate over the boundary of the unit circle γ.

$$\frac{\alpha}{2\pi i}\int_{\gamma}\frac{\phi(\sigma)}{\sigma-\xi}\,d\sigma + \frac{1}{2\pi i}\int_{\gamma}\frac{W(\sigma)}{\overline{W'(\sigma)}}\,\frac{\overline{\phi'(\sigma)}}{\sigma-\xi}\,d\sigma + \frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\psi(\sigma)}}{\sigma-\xi}\,d\sigma$$

$$= A(\xi)\ , \tag{5.7}$$

where

$$A(\xi) = \frac{1}{2\pi i}\int_{\gamma}\frac{H(\sigma)}{\sigma-\xi}\,d\sigma\ .$$

We obtain the following integro-differential equation in $\phi(\xi)$

$$\alpha\,\phi(\xi) + \frac{1}{2\pi i}\int_{\gamma}\frac{W(\sigma)}{\overline{W'(\sigma)}}\,\frac{\overline{\phi'(\sigma)}}{\sigma-\xi}\,d\sigma + \overline{\psi(o)} = A(\xi)\ . \tag{5.8}$$

It contains an unknown constant $\overline{\psi(o)}$ which can be determined by imposing the condition $\phi(o) = 0$. Having determined $\phi(\xi)$, the value of $\psi(\xi)$ is determined by forming the conjugate of (5.6).

$$\psi(\sigma) = \overline{H(\sigma)} - \frac{\overline{W(\sigma)}}{W'(\sigma)}\,\phi'(\sigma) - \alpha\,\overline{\phi(\sigma)}\ .$$

Multiplying the above equation by $\frac{1}{2\pi i}\,\frac{d\sigma}{\sigma-\xi}$ and integrating over γ

we get

$$\psi(\xi) = \frac{1}{2\pi i} \int_{\gamma} \frac{\overline{H(\sigma)}}{\sigma - \xi} d\sigma - \frac{1}{2\pi i} \int_{\gamma} \frac{\overline{W(\sigma)}}{\overline{W'(\sigma)}} \frac{\overline{\phi'(\sigma)}}{\sigma - \xi} d\sigma \qquad (5.9)$$

which gives $\psi(\xi)$.

We follow the above mentioned procedure for the equation (5.5) and obtain

$$\phi_{1o}(\xi) + \frac{1}{2\pi i} \int_{\gamma} \frac{\sigma \overline{\phi'_{1o}(\sigma)}}{\sigma - \xi} d\sigma + \overline{\psi_{1o}(0)} = \frac{3+\nu}{8} \rho \omega^2 R^3 \xi$$

$$+ \frac{4\mu}{k+1} \left\{ \frac{\delta_1 a_1^2 \xi}{R - d_1 \xi} + \frac{\delta_2 a_2^2 \xi}{R - d_2 \xi} \right\}. \qquad (5.10)$$

Since $\phi_{1o}(\xi)$ is analytic in the unit circle, it has the power series expansion,

$$\phi_{1o}(\xi) = \sum_{n=1}^{\infty} a_n \xi^n, \quad \overline{\phi'_{1o}(\xi)} = \sum_{n=1}^{\infty} n \bar{a}_n \xi^{-(n-1)} .$$

Substituting for $\phi_{1o}(\xi)$, $\overline{\phi'_{1o}(\xi)}$ in the equation (5.10), we obtain

$$\sum_{n=1}^{\infty} a_n \xi^n + \bar{a}_1 \xi + 2\bar{a}_2 + \overline{\psi(o)} = \frac{3+\nu}{8} \rho \omega^2 R^3 \xi + \frac{4\mu}{k+1} \left[\frac{\delta_1 a_1^2 \xi}{R - d_1 \xi} + \frac{\delta_2 a_2^2 \xi}{R - d_2 \xi} \right].$$

By equating the like powers of ξ on both sides, it may be easily seen that

$$2\bar{a}_2 + \overline{\psi(o)} = 0$$

and

$$a_1 = \frac{3+\nu}{16} \rho \omega^2 R^3 + \frac{2\mu}{k+1} \left(\frac{\delta_1 a_1^2}{R} + \frac{\delta_2 a_2^2}{R} \right) .$$

Hence

$$\phi_{1o}(\xi) = \frac{3+\nu}{16} \rho \omega^2 R^3 \xi + \frac{2\mu}{k+1} \left\{ \frac{\delta_1 a_1^2 \xi (R+d_1\xi)}{R(R - d_1\xi)} + \frac{\delta_2 a_2^2 \xi (R+d_2\xi)}{R(R - d_2\xi)} \right\} . \tag{5.11}$$

As described above, the value of $\psi_{1o}(\xi)$ is obtained from the equation

$$\psi_{1o}(\xi) = \frac{1}{2\pi i} \int_\gamma \frac{\overline{H(\sigma)}}{\sigma - \xi} d\sigma - \frac{1}{2\pi i} \int_\gamma \frac{\overline{W(\sigma)}}{W'(\sigma)} \frac{\overline{\phi_{1o}'(\sigma)}}{\sigma - \xi} d\sigma$$

where

$$\overline{H(\sigma)} = \frac{3+\nu}{8} \rho \omega^2 R^3 \frac{1}{\sigma} + \frac{4\mu}{k+1} \left\{ \frac{\delta_1 a_1^2}{R\sigma - d_1} + \frac{\delta_2 a_2^2}{R\sigma - d_2} \right\} .$$

It may be easily seen that

$$\psi_{1o}(\xi) = - \frac{4\mu}{k+1} \left[\left\{ \frac{\delta_1 d_1 a_1^2}{R} \frac{(2R - d_1\xi)}{(R - d_1\xi)^2} \right\} + \frac{\delta_2 d_2 a_2^2}{R} \frac{(2R - d_2\xi)}{(R - d_2\xi)^2} \right] . \tag{5.12}$$

Substituting $\xi = \frac{z}{R}$, it is observed that

$$\phi_o(z) = \frac{3+\nu}{16} \rho \omega^2 R^2 z + \frac{2\mu}{k+1} \left\{ \frac{\delta_1 a_1^2 z (R^2 + d_1 z)}{R^2(R^2 - d_1 z)} + \frac{\delta_2 a_2^2 z (R^2 + d_2 z)}{R^2(R^2 - d_2 z)} \right\} ,$$

$$\psi_o(z) = -\frac{4\mu}{k+1}\left\{\frac{\delta_1 d_1 a_1^2(2R^2 - d_1 z)}{(R^2 - d_1 z)^2} + \frac{\delta_2 d_2 a_2^2(2R^2 - d_2 z)}{(R^2 - d_2 z)^2}\right\}.$$

Therefore,

for any point z in R_{11},

$$\phi_{11}(z) = \frac{3+\nu}{16}\rho\omega^2 R^2 z + \frac{2\mu\delta_1}{k+1}\frac{a_1^2 z(R^2 + d_1 z)}{R^2(R^2 - d_1 z)} + \frac{2\mu\delta_2}{k+1} \times \frac{a_2^2 z(R^2 + d_2 z)}{R^2(R^2 - d_2 z)} - \frac{2\mu\delta_1}{k+1}(z - d_1),$$

$$\psi_{11}(z) = -\frac{4\mu}{k+1}\left[\frac{\delta_1 d_1 a_1^2(2R^2 - d_1 z)}{(R^2 - d_1 z)^2} + \frac{\delta_2 d_2 a_2^2(2R^2 - d_2 z)}{(R^2 - d_2 z)^2} + \frac{\delta_2 a_2^2}{z - d_2} - \frac{\delta_1 d_1}{2}\right];$$

for any point z in R_{12},

$$\phi_{12}(z) = \frac{3+\nu}{16}\rho\omega^2 R^2 z + \frac{2\mu\delta_1}{k+1}\frac{a_1^2 z(R^2 + d_1 z)}{(R^2 - d_1 z)R^2} + \frac{2\mu\delta_2}{k+1}\frac{a_2^2 z(R^2 + d_2 z)}{R^2(R^2 - d_2 z)} - \frac{2\mu\delta_2}{k+1}(z - d_2),$$

$$\psi_{12}(z) = -\frac{4\mu}{k+1}\left[\frac{\delta_1 d_1 a_1^2(2R^2 - d_1 z)}{(R^2 - d_1 z)^2} + \frac{\delta_2 d_2 a_2^2(2R^2 - d_2 z)}{(R^2 - d_2 z)^2} + \frac{\delta_1 a_1^2}{z - d_1} + \frac{\delta_2 d_2}{2}\right];$$

and for any point z in R_s ,

$$\phi_s(z) = \frac{3+\nu}{16} \rho \omega^2 R^2 z + \frac{2\mu\delta_1}{k+1} \frac{a_1^2 z(R^2 + d_1 z)}{R^2(R^2 - d_1 z)}$$

$$+ \frac{2\mu\delta_2}{k+1} \frac{a_2^2 z(R^2 + d_2 z)}{R^2(R^2 - d_2 z)} ,$$

$$\psi_s(z) = -\frac{4\mu}{k+1} \left[\frac{\delta_1 d_1 a_1^2 (2R^2 - d_1 z)}{(R^2 - d_1 z)^2} + \frac{\delta_2 d_2 a_2^2 (2R^2 - d_2 z)}{(R^2 - d_2 z)^2} \right.$$

$$\left. + \frac{\delta_1 a_1^2}{z - d_1} + \frac{\delta_2 a_2^2}{z - d_2} \right] .$$

The stresses and displacements for the homogeneous problem may be evaluated at any point using the formulae

$$p_{rr}^{(2)} + p_{\theta\theta}^{(2)} = 4 \operatorname{Re} \phi'(z) ,$$

$$p_{\theta\theta}^{(2)} - p_{rr}^{(2)} + 2i p_{r\theta}^{(2)} = 2e^{2i\theta} \left[\bar{z} \phi''(z) + \psi'(z) \right] ,$$

$$2\mu \left(u_r^{(2)} + i u_\theta^{(2)} \right) = e^{-i\theta} \left\{ k \phi(z) - z \overline{\phi'(z)} - \overline{\psi(z)} \right\}$$

where $\phi(z) = \phi_o(z) + \phi_*(z)$ and $\psi(z) = \psi_o(z) + \psi_*(z)$. To get the actual stresses $p_{\alpha\beta}$ $(\alpha,\beta = r,\theta)$ and displacements u_α, one has to add $p_{\alpha\beta}^{(1)}$, $u_\alpha^{(1)}$ $(\alpha,\beta = r,\theta)$ given by (2.1) and (2.2) , to $p_{\alpha\beta}^{(2)}$, $u_\alpha^{(2)}$.

At the equilibrium interface L_1, one may easily see that

$$(P_{rr}^{i1} - P_{rr}^{s}) - i(P_{r\theta}^{i1} - P_{r\theta}^{s}) = 2 \operatorname{Re} \left\{ \phi_*'{}_{i1}(z) - \phi_*'{}_{s}(z) \right\}$$

$$- e^{2i\theta} \left[\bar{z} \left(\phi_*''{}_{i1}(z) - \phi_*''{}_{s}(z) \right) + \psi_*'{}_{i1}(z) - \psi_*'{}_{s}(z) \right]$$

$$= - \frac{4\mu\delta_1}{k + 1} + \frac{4\mu}{k + 1} \quad \frac{\delta_1 a^2}{(z - d_1)^2} \; e^{2i\theta} = 0$$

since $z - d_1 = ae^{i\theta}$ on L_1, which implies that $P_{rr}^{i1} = P_{rr}^{s}$ and $P_{r\theta}^{i1} = P_{r\theta}^{s}$ as expected. Similarly it may be proved that the normal and shearing stresses are continuous at the other equilibrium boundary L_2.

The jump in hoop stress at the equilibrium boundary L_1 is given by

$$P_{\theta\theta}^{i1} - P_{\theta\theta}^{s} = - \frac{8\mu\delta_1}{k + 1} .$$

Some particular cases, when there is only one insert, or when the centre of an insert is at the centre of the outer disc, may be evaluated by giving particular values to δ_1, δ_2 and d_1, d_2. Similarly, the effect of rotation may be neglected by taking $\omega = 0$. The results in all these particular cases agree with all those which are obtained.

CHAPTER 6

Rotating Disc With An Eccentric Elliptic Insert

In this chapter, the problem of an elliptic inclusion eccentrically situated in a rotating shell is considered. The inclusion undergoes a spontaneous homogeneous deformation, which in the absence of the shell is prescribed. The composite disc is rotating about an axis through the centre of the disc perpendicular to its plane. The complex potentials which give the elastic field in the insert and the shell suffer a discontinuity whenever they cross the common interface between the shell and the insert. Because of the misfit in size of the insert and also because of rotation, stresses develop both in the insert and in the shell. The problem is solved with the help of the Hilbert problem discussed in the last chapter. Two sets of complex potentials, one for the insert and another for the shell, are evaluated to find the elastic field everywhere.

Consider a circular disc of radius R with an eccentric elliptic region of semi-major axis a and semi-minor axis b. The outer boundary of the shell is denoted by L_0 and the inner boundary is denoted by L. The centre O, of the shell, is taken as the origin and the line joining O and the geometrical centre of the elliptic region is taken as the X-axis. The distance between the

two centres is d, ($|d| < R - a$). Now this elliptic region, called the insert, in the absence of the shell would have undergone a spontaneous homogeneous deformation defined by $u = \delta_1(x-d)$, $v = \delta_2 y$; δ_1 and δ_2 lie within elastic limits. Because of the constraints of the outer region, the shell, the stresses develop both in the insert and in the shell. The composite disc is rotating about an axis through the origin O perpendicular to its plane, with a constant angular velocity ω .

Let the region occupied by the insert be referred to as R_i and that of the shell by R_s. The holomorphic functions are distinguished in these regions by subscripts i and s respectively. It is stated in the last chapter that if $F(z)$ is a holomorphic function in the whole region R ($= R_i \cup R_s$) and if the discontinuity of $F(z)$ on L is defined by $L(t)$ i.e., if

$$F^+(t) - F^-(t) = L(t) \quad \text{on } L$$

then

$$F(z) = F^*(z) + \frac{1}{2\pi i} \int_L \frac{L(t)}{t - z} \, dt$$

where t is any point on L and $F^*(z)$ is the function analytic in the whole region R, evaluated from the boundary conditions.

Let (u_i, v_i) be the displacements components of the insert and (u_s, v_s) be those of the shell. At the equilibrium interface, if the displacement components be denoted by (u_s^b, v_s^b) and (u_i^b, v_i^b), then

$$u_1^b - u_s^b = -\delta_1\left(\frac{t+\bar{t}}{2} - d\right) = g_1(t) ,$$

$$v_1^b - v_s^b = -\delta_2\left(\frac{t-\bar{t}}{2i}\right) = g_2(t) .$$

It may be remarked here that (u_1^b , v_1^b), the displacements at the boundary of the insert, is measured from its natural state : the state it would have in the absence of the shell; while (u_s^b , v_s^b) is the displacement at the boundary of the shell measured from its natural state, which is the state in the absence of the insert before rotation.

Let $\phi(z)$, $\psi(z)$ be the holomorphic functions which give the stress field, when the body forces are absent. Because of the discontinuity at L, the functions

$$\phi(z) = \phi_0(z) + \phi_*(z) , \quad (z) = \psi_0(z) + \psi_*(z) \tag{6.1}$$

where $\{\phi_0(z), \psi_0(z)\}$ are functions analytic in the whole region and $\{\phi_*(z), \psi_*(z)\}$ are given by

$$\phi_*(z) = \frac{\mu}{k+1}\,\frac{1}{\pi i}\int_L \frac{g_1(t) + i\,g_2(t)}{t-z}\,dt ,$$

$$\psi_*(z) = \frac{\mu}{k+1}\,\frac{1}{\pi i}\int_L - \frac{\left(\overline{g_1(t)} + i\,\overline{g_2(t)} + \bar{t}\,g_1'(t) + i\,\bar{t}\,g_2'(t)\right)}{t-z}\,dt .$$

In the present case

$$g_1(t) + i\, g_2(t) = -\left(\frac{\delta_1 + \delta_2}{2} t + \frac{\delta_1 - \delta_2}{2} \bar{t} - \delta_1 d \right) ,$$

and

$$-\left(\overline{g_1(t)} + i\, \overline{g_2(t)} + \bar{t}\, g_1'(t) + i\, \bar{t}\, g_2'(t)\right) = (\delta_1+\delta_2)\bar{t} + \frac{(\delta_1-\delta_2)}{2} t - \delta_1 d + \frac{(\delta_1-\delta_2)}{2} \bar{t}\ \frac{d\bar{t}}{dt}$$

where $\bar{t}$ is given by

$$\bar{t} = \frac{a^2 + b^2}{a^2 - b^2} t - \frac{2b^2 d}{a^2-b^2} - \frac{2ab}{a^2- b^2} \sqrt{(t-d)^2 - (a^2- b^2)} \ .$$

This follows from the fact that the equation of the ellipse of semiaxes a and b and centre $(d,0)$ is $((x-d)^2/a^2)+(y^2/b^2) = 1$. Putting $x = (t + \bar{t})/2$ and $y = (t - \bar{t})/2i$ and solving for $\bar{t}$, we get the above expression. The sign before the squareroot bracket is obtained by the consideration that at $(a + d, 0)$ and $(a - d, 0)$, $t = \bar{t}$. For abreviation, we shall put

$$c^2 = a^2 - b^2 \ .$$

It may be easily seen that in the case of the insert

$$\phi_*(z) = -\frac{\mu}{k + 1} \left[(\delta_1+\delta_2)z - 2\delta_1 d +(\delta_1-\delta_2)\left(\frac{a-b}{a+b} z + \frac{2bd}{a + b} \right)\right] ,$$

$$\psi_*(z) = \frac{\mu}{k+1}\left[(\delta_1-\delta_2)z - 2\delta_1 d + 2(\delta_1+\delta_2)\left(\frac{a-b}{a+b}z + \frac{2bd}{a+b}\right)\right.$$

$$\left. + (\delta_1-\delta_2)\left\{\frac{(a-b)^2}{(a+b)^2}z + \frac{2bd(a-b)}{(a+b)^2}\right\}\right]$$

(6.2)

and for the shell

$$\phi_*(z) = \frac{\mu}{k+1}\,\frac{2ab}{a^2-b^2}\,(\delta_1-\delta_2)\left\{(z-d) - \sqrt{(z-d)^2 - c^2}\right\},$$

$$\psi_*(z) = -\frac{\mu}{k+1}\,\frac{2ab}{a^2-b^2}\left[\,2(\delta_1+\delta_2)\left\{(z-d) - \sqrt{(z-d)^2 - c^2}\right\}\right.$$

$$+ (\delta_1-\delta_2)\frac{a^2+b^2}{a^2-b^2}\left\{2z - \frac{z(z-d)}{\sqrt{(z-d)^2-c^2}} - d - \sqrt{(z-d)^2-c^2}\right\}$$

$$\left. - (\delta_1-\delta_2)\frac{2b^2d}{a^2-b^2}\left\{1 - \frac{z-d}{\sqrt{(z-d)^2-c^2}}\right\}\right].$$

(6.3)

We evaluate $\phi_o(z)$, $\psi_o(z)$ by the same conditions as in chapter 5. The boundary conditions satisfied on L_o is

$$\phi_o(t) + t\,\overline{\phi_o'(t)} + \overline{\psi_o(t)} = f_o(t)$$

where

$$f_o(t) = f^{(2)}(t) - \phi_*(t) - t\,\overline{\phi_*'(t)} - \overline{\psi_*(t)}$$

and

$$f^{(2)}(t) = i \int^{s} \left\{ (P_{nx} - P_{nx}^{(1)}) + i\,(P_{ny} - P_{ny}^{(1)}) \right\} ds$$

where $P_{n\alpha}$ $(\alpha = x,y)$ is the given stress vector on the outer boundary; $P_{n\alpha}^{(1)}$ $(\alpha = x,y)$ is the induced stress vector on the boundary because of the rotation. Note that $P_{n\alpha}^{(1)}$ $(\alpha = x,y)$ are given by the formula (1.18). In the present case, the total traction $P_{n\alpha}$ $(\alpha = x,y)$ is zero since the outer boundary is free from stress and $f^{(2)}(t) = \frac{3+\nu}{8} \rho \omega^2 R^2 t$, where t is a point on L_o. Therefore the boundary condition on L_o reduces to

$$\phi_o(t) + t\,\overline{\phi_o'(t)} + \overline{\psi_o(t)} = \frac{3+\nu}{8} \rho \omega^2 R^2 t - \frac{\mu}{k+1} \frac{2ab}{a^2-b^2} \Big[(\delta_1-\delta_2)$$

$$\left\{ (t-d) - \sqrt{(t-d)^2 - e^2} \right\} + (\delta_1-\delta_2) \left\{ t - \frac{R^2 - d\bar{t}}{\sqrt{(\bar{t}-d)^2 - e^2}} \right\}$$

$$- 2(\delta_1+\delta_2) \left\{ (\bar{t}-d) - \sqrt{(\bar{t}-d)^2 - e^2} \right\} - (\delta_1-\delta_2) \frac{a^2+b^2}{a^2-b^2}$$

$$\left\{ 2\bar{t} - \frac{\bar{t}\,(\bar{t}-d)}{\sqrt{(\bar{t}-d)^2 - e^2}} - d - \sqrt{(\bar{t}-d)^2 - e^2} \right\}$$

$$+ (\delta_1-\delta_2) \frac{2b^2 d}{a^2-b^2} \left\{ 1 - \frac{\bar{t}-d}{\sqrt{(\bar{t}-d)^2 - e^2}} \right\} \Big].$$

The region R is mapped to the unit circle $|\xi| < 1$ by the transformation $z = R\,\xi$. Then the above equation takes the form

$$
\begin{aligned}
\phi_{1o}(\sigma) + \sigma\,\overline{\phi'_{1o}(\sigma)} + \overline{\psi_{1o}(\sigma)} = {} & \frac{3+\nu}{8}\,\rho\,\omega^2 R^3 \sigma \; - \frac{\mu}{\kappa+1}\,\frac{2ab}{a^2-b^2} \\
& \Bigg[(\delta_1-\delta_2) \left\{ (R\sigma - d) - \sqrt{(R\sigma - d)^2 - c^2} \right\} + (\delta_1-\delta_2) \Bigg\{ R\sigma \\
& - \frac{(R^2 - Rd\sigma)\sigma}{\sqrt{(R-d\sigma)^2 - c^2\sigma^2}} \Bigg\} - \Bigg\{ 2(\delta_1+\delta_2) \Bigg\{ \frac{(R - d\sigma)}{\sigma} \\
& - \frac{\sqrt{(R-d\sigma)^2 - c^2\sigma^2}}{\sigma} \Bigg\} + (\delta_1-\delta_2)\,\frac{(a^2+b^2)}{a^2-b^2} \Bigg\{ \frac{2R}{\sigma} - d \\
& - \frac{R(R - d\sigma)}{\sigma\sqrt{(R-d\sigma)^2 - c^2\sigma^2}} - \frac{\sqrt{(R-d\sigma)^2 - c^2\sigma^2}}{\sigma} \Bigg\} \\
& - (\delta_1-\delta_2)\,\frac{2b^2 d}{a^2-b^2} \left\{ 1 - \frac{(R - d\sigma)}{\sqrt{(R-d\sigma)^2 - c^2\sigma^2}} \right\} \Bigg\} \Bigg]
\end{aligned}
\tag{6.4}
$$

where σ is a point on the unit circle γ. The integro-differential equation technique used and stated in chapter 5 is used to evaluate $\phi_{1o}(\xi)$. It turns out to be that

$$\phi_{1o}(\xi) = \frac{3+\nu}{16}\rho\,\omega^2 R^3 \xi - \frac{\mu}{k+1}\frac{2ab}{a^2-b^2}(\delta_1-\delta_2)\left\{R\xi - \frac{(R^2-Rd\xi)\xi}{\sqrt{(R-d\xi)^2-c^2\xi^2}}\right\}$$

$$+ \frac{\mu}{k+1}\frac{2ab}{a^2-b^2}\Bigg[2(\delta_1+\delta_2)\left\{\frac{R-d\xi}{\xi} - \frac{\sqrt{(R-d\xi)^2-c^2\xi^2}}{\xi}\right\}$$

$$+ (\delta_1-\delta_2)\frac{a^2+b^2}{a^2-b^2}\left\{\frac{2R}{\xi} - d - \frac{R(R-d\xi)}{\xi\sqrt{(R-d\xi)^2-c^2\xi^2}} - \frac{\sqrt{(R-d\xi)^2-c^2\xi^2}}{\xi}\right\}$$

$$- (\delta_1-\delta_2)\frac{2b^2d}{a^2-b^2}\left\{1 - \frac{(R-d\xi)}{\sqrt{(R-d\xi)^2-c^2\xi^2}}\right\} - \frac{c^2(\delta_1+\delta_2)\xi}{2R}\Bigg] .$$

(6.5)

The function $\psi_{1o}(\xi)$ is given by

$$\psi_{1o}(\xi) = \frac{1}{2\pi i}\int_{\gamma}\frac{\overline{H(\sigma)}}{\sigma-\xi}\,d\sigma - \frac{1}{2\pi i}\int_{\gamma}\frac{\overline{W(\sigma)}}{W'(\sigma)}\,\frac{\phi'_{1o}(\sigma)}{\sigma-\xi}\,d\sigma$$

(6.6)

where γ is the unit circle, $W(\xi) = R\xi$ and

$$\overline{H(\sigma)} = \frac{3+\nu}{8}\rho\,\omega^2 R^3\frac{1}{\sigma} - \frac{\mu}{k+1}\frac{2ab}{a^2-b^2}(\delta_1-\delta_2)\left\{\frac{R}{\sigma} - d\right.$$

$$\left. - \frac{\sqrt{(R-d\sigma)^2-c^2\sigma^2}}{\sigma}\right\} - \frac{\mu}{k+1}\frac{2ab}{a^2-b^2}(\delta_1-\delta_2)\Bigg[\left\{\frac{R}{\sigma} - \frac{(R^2\sigma-Rd)}{\sigma\sqrt{(R\sigma-d)^2-c^2}}\right\}$$

$$- 2(\delta_1+\delta_2)\left\{(R\sigma-d) - \sqrt{(R\sigma-d)^2-c^2}\right\} - (\delta_1-\delta_2)\frac{a^2+b^2}{a^2-b^2}\left\{2R\sigma\right.$$

$$\left. - \frac{R(R\sigma-d)\sigma}{\sqrt{(R\sigma-d)^2-c^2}} - d - \sqrt{(R\sigma-d)^2-c^2}\right\} + (\delta_1-\delta_2)\frac{2b^2d}{a^2-b^2} \quad x$$

$$\left\{1 - \frac{(R\sigma - d)}{\sqrt{(R\sigma - d)^2 - c^2}}\right\}\Bigg] . \qquad (6.7)$$

Since $\overline{H(\sigma)}$ and $\overline{\phi'_{1o}(\sigma)}$ are known, $\psi_{1o}(\xi)$ can be calculated from the equation (6.6). Substituting $\xi = \frac{z}{R}$ in $\phi_{1o}(\xi)$, $\psi_{1o}(\xi)$ one can find $\{\phi_o(z),\ \psi_o(z)\}$ as follows :

$$\phi_o(z) = \frac{3+\nu}{16}\rho\omega^2R^2z - \frac{\mu}{k+1}\frac{2ab}{a^2-b^2}(\delta_1-\delta_2)\left\{z - \frac{(R^2-dz)\,z}{\sqrt{(R^2-dz)^2-c^2z^2}}\right\}$$

$$+\frac{\mu}{k+1}\frac{2ab}{a^2-b^2}\Bigg[2(\delta_1+\delta_2)\left\{\frac{R^2-dz}{z} - \frac{\sqrt{(R^2-dz)^2-c^2z^2}}{z}\right\}+(\delta_1-\delta_2)$$

$$\left(\frac{a^2+b^2}{a^2-b^2}\right)\left\{\frac{2R^2}{z} - d - \frac{R^2(R^2-dz)}{z\sqrt{(R^2-dz)^2-c^2z^2}} - \frac{\sqrt{(R^2-dz)^2-c^2z^2}}{z}\right\}$$

$$-(\delta_1-\delta_2)\frac{2b^2d}{a^2-b^2}\left\{1 - \frac{(R^2-dz)}{\sqrt{(R^2-dz)^2-c^2z^2}}\right\}\Bigg] - \frac{\mu}{k+1}\frac{ab}{R^2}(\delta_1+\delta_2)z ,$$

$$\psi_o(z) = \frac{\mu}{k+1}\frac{2ab}{a^2-b^2}\Bigg[(\delta_1-\delta_2)\left\{\frac{\sqrt{(R^2-dz)^2-c^2z^2}-(R^2-dz)}{z}\right\}-(\delta_1-\delta_2)$$

$$\left\{\frac{R^2(R^2-dz)^3}{\{(R^2-dz)^2-c^2z^2\}^{3/2}}\frac{1}{z} + \frac{c^2R^2dz^2}{\{(R^2-dz)^2-c^2z^2\}^{3/2}} - \frac{R^2}{z}\right\}$$

$$-2(\delta_1+\delta_2)\left\{\frac{R^4(R^2-dz)}{\sqrt{(R^2-dz)^2-c^2z^2}}\frac{1}{z^3} - \frac{c^2}{2z} - \frac{R^4}{z^3}\right\} - (\delta_1-\delta_2)\frac{a^2+b^2}{a^2-b^2}$$

$$\frac{R^4(R^2-dz)^3}{z^3\left\{(R^2-dz)^2-c^2z^2\right\}^{3/2}}-\frac{2R^4}{z^3}-\frac{R^4c^2(2R^2-dz)}{\left\{(R^2-dz)^2-c^2z^2\right\}^{3/2}}\;\frac{1}{z}$$

$$+\frac{R^4(R^2-dz)}{z^3\left\{(R^2-dz)^2-c^2z^2\right\}^{1/2}}-(\delta_1-\delta_2)\,\frac{2b^2d}{a^2-b^2}\;\frac{R^4c^2}{\left\{(R^2-dz)^2-c^2z^2\right\}^{3/2}}\Bigg]\,.$$

(6.8)

Since $\{\phi_*(z),\psi_*(z)\}$, $\{\phi_0(z),\psi_0(z)\}$ are known functions from (6.2), (6.3) and (6.8), we substitute them back in (6.1) and obtain the complex potentials for the insert as

$$\phi_1(z)=-\frac{\mu}{k+1}\left[(\delta_1+\delta_2)z-2\delta_1 d+(\delta_1-\delta_2)\left(\frac{a-b}{a+b}z+\frac{2bd}{a+b}\right)\right]+\phi_0(z)\,,$$

$$\psi_1(z)=\frac{\mu}{k+1}\Bigg[(\delta_1-\delta_2)z-2\delta_1 d+2(\delta_1+\delta_2)\left(\frac{a-b}{a+b}z+\frac{2bd}{a+b}\right)$$

$$+(\delta_1-\delta_2)\left\{\frac{(a-b)^2}{(a+b)^2}z+\frac{2bd}{(a+b)^2}(a-b)\right\}\Bigg]+\psi_0(z)\,.$$

(6.9)

For the shell

$$\phi_s(z)=\frac{\mu}{k+1}\,\frac{2ab}{a^2-b^2}\,(\delta_1-\delta_2)\left\{(z-d)-\sqrt{(z-d)^2-c^2}\right\}+\phi_0(z)\,,$$

$$\psi_s(z)=-\frac{\mu}{k+1}\,\frac{2ab}{a^2-b^2}\Bigg[2(\delta_1+\delta_2)\left\{(z-d)-\sqrt{(z-d)^2-c^2}\right\}$$

$$+(\delta_1-\delta_2)\,\frac{a^2+b^2}{a^2-b^2}\left\{2z-\frac{z(z-d)}{\sqrt{(z-d)^2-c^2}}-d-\sqrt{(z-d)^2-c^2}\right\}$$

$$-(\delta_1-\delta_2)\,\frac{2b^2d}{a^2-b^2}\left\{1-\frac{(z-d)}{\sqrt{(z-d)^2-c^2}}\right\}\Bigg]+\psi_0(z)\,.$$

(6.10)

This completely solves the problem. One can for example, evaluate the stress and displacement field, for the homogeneous problem, in the insert and in the shell using these complex potentials and the formula (1.19). The actual stresses, for the non-homogeneous problem may be obtained by adding the stresses given by (1.15) to those obtained in the homogeneous problem.

It is however instructive to find the normal and tangential stresses at the equilibrium interface. For this, it is simpler to transfer the origin (d,0). The potential functions $\{\phi_1(z'), \psi_1(z')\}$ become

$$\phi_1(z') = \phi(z' + d) \quad \text{and} \quad \psi_1(z') = \psi(z' + d) + d\,\phi'(z' + d)$$

where $\{\phi_1(z'), \psi_1(z')\}$ are the potential functions referred to (d,0) as the origin. This follows from the formula (1.26) stated in chapter 1.

It is again simpler to write $z = c \cosh \zeta$ where $\zeta = \xi + i\eta$. Note that ξ = constant gives confocal ellipses of semiaxes $c \cosh \xi$, $c \sinh \xi$ and η = constant gives hyperbolas. Let the elliptic boundary be given by $\xi = \xi_0$. Then

$$c \cosh \xi_0 = a \quad \text{and} \quad c \sinh \xi_0 = b\,.$$

Using the obvious notation $P_{\xi\xi}$, $P_{\eta\eta}$ to the normal stresses in the increasing directions of ξ, η respectively and $P_{\xi\eta}$ to be the

shear stress, it may be seen that

$$P_{\xi\xi} + P_{\eta\eta} = P_{xx} + P_{yy} \ ,$$

$$P_{\eta\eta} - P_{\xi\xi} + 2i\, P_{\xi\eta} = e^{2i\alpha}\,(P_{yy} - P_{xx} + 2i\, P_{xy}) \ , \tag{6.11}$$

where α is the angle which the normal ξ = constant makes with the X-axis. Similarly, if u_ξ, u_η are the displacement components in the ξ, η directions, then

$$u_\xi + i\, u_\eta = e^{-i\alpha}\,(u_x + i\, u_y) \ .$$

Thus from (1.19) and (6.11)

$$p^{(2)}_{\xi\xi} + p^{(2)}_{\eta\eta} = 4\, \mathrm{Re}\, \phi_1'(z') \ ,$$

$$p^{(2)}_{\eta\eta} - p^{(2)}_{\xi\xi} + 2i\, p^{(2)}_{\xi\eta} = 2e^{2i\alpha}\,(\bar{z}'\, \phi_1''(z') + \psi_1'(z')) \ ,$$

$$2\mu\,(u^{(2)}_\xi + i\, u^{(2)}_\eta) = e^{-i\alpha}\,(k\,\phi_1(z') - \bar{z}'\,\overline{\phi_1'(z')} - \overline{\psi_1(z')}) \ . \tag{6.12}$$

In the case of insert

$$p^{(2)}_{\xi\xi} - i\, p^{(2)}_{\xi\eta} = \phi_*'(z'+d) + \overline{\phi_*'(z'+d)} - e^{2i\alpha}\Big\{\bar{z}'\, \phi_*''(z'+d) + d\,\phi_*''(z'+d) + \psi_*'(z'+d)\Big\} + \phi_0'(z'+d) + \overline{\phi_0'(z'+d)} - e^{2i\alpha}\Big\{\bar{z}'\,\phi_0''(z'+d) + d\,\phi_0''(z'+d) + \psi_0'(z'+d)\Big\} \ .$$

Therefore

$$p^{(2)}_{\xi\xi} - i\, p^{(2)}_{\xi\eta} = -\frac{\mu}{k+1}\Bigg[\frac{4ab}{a+b}(\delta_1+\delta_2)\frac{\cos\eta - i\sin\eta}{b\cos\eta - ia\sin\eta}$$

$$+\frac{4ab}{(a+b)^2}\;\frac{(\delta_1-\delta_2)(a\cos\eta + ib\sin\eta)}{c\sinh\bar{\xi}} + \phi_o'(z'+d)$$

$$+\overline{\phi_o'(z'+d)} - e^{2i\alpha}\left(\bar{z}'\phi_o''(z'+d) + d\,\phi_o''(z'+d) + \psi_o'(z'+d)\right)\Bigg].$$

(6.13)

Similarly for the shell

$$p^{(2)}_{\xi\xi} - i\, p^{(2)}_{\xi\eta} = -\frac{\mu}{k+1}\Bigg[\frac{4ab}{a+b}(\delta_1+\delta_2)\;\frac{\cos\eta - i\sin\eta}{b\cos\eta - ia\sin\eta}$$

$$+\frac{4ab}{(a+b)^2}(\delta_1-\delta_2)\;\frac{a\cos\eta + ib\sin\eta}{c\sinh\bar{\xi}} + \phi_o'(z'+d)$$

$$+\overline{\phi_o'(z'+d)} - e^{2i\alpha}\left(\bar{z}'\phi_o''(z'+d) + d\,\phi_o''(z'+d) + \psi_o'(z'+d)\right)\Bigg].$$

(6.14)

From (6.13) and (6.14) we observe that the normal and shearing stresses are continuous on the equilibrium boundary, as it should be.

The particular case of a composite circular disc with a concentric elliptic insert can be derived from the earlier results by putting $d = 0$ and $\omega = 0$. Then, for the insert

Therefore

$$p^{(2)}_{\xi\xi} - i\, p^{(2)}_{\xi\eta} = -\frac{\mu}{k+1}\left[\frac{4ab}{a+b}(\delta_1+\delta_2)\frac{\cos\eta - i\sin\eta}{b\cos\eta - ia\sin\eta}\right.$$

$$+ \frac{4ab}{(a+b)^2}\quad\frac{(\delta_1-\delta_2)(a\cos\eta + ib\sin\eta)}{c\sinh\bar{\xi}} + \phi_o'(z'+d)$$

$$\left. + \overline{\phi_o'(z'+d)} - e^{2i\alpha}\left(\bar{z}'\phi_o''(z'+d) + d\,\phi_o''(z'+d) + \psi_o'(z'+d)\right)\right].$$

(6.13)

Similarly for the shell

$$p^{(2)}_{\xi\xi} - i\, p^{(2)}_{\xi\eta} = -\frac{\mu}{k+1}\left[\frac{4ab}{a+b}(\delta_1+\delta_2)\frac{\cos\eta - i\sin\eta}{b\cos\eta - ia\sin\eta}\right.$$

$$+ \frac{4ab}{(a+b)^2}(\delta_1-\delta_2)\quad\frac{a\cos\eta + ib\sin\eta}{c\sinh\bar{\xi}} + \phi_o'(z'+d)$$

$$\left. + \overline{\phi_o'(z'+d)} - e^{2i\alpha}\left(\bar{z}'\phi_o''(z'+d) + d\,\phi_o''(z'+d) + \psi_o'(z'+d)\right)\right].$$

(6.14)

From (6.13) and (6.14) we observe that the normal and shearing stresses are continuous on the equilibrium boundary, as it should be.

The particular case of a composite circular disc with a concentric elliptic insert can be derived from the earlier results by putting $d = 0$ and $\omega = 0$. Then, for the insert

$$\phi_1(z) = -\frac{\mu}{k+1}\left\{(\delta_1+\delta_2)z+(\delta_1-\delta_2)\frac{a-b}{a+b}z\right\}+\frac{\mu}{k+1}\frac{2ab}{a^2-b^2}\quad x$$

$$\left[(\delta_1-\delta_2)\left(\frac{R^2}{\sqrt{R^4-c^2z^2}}-1\right)z-2(\delta_1+\delta_2)\left\{\sqrt{R^4-c^2z^2}-R^2+\frac{c^2z^2}{4R^2}\right\}\frac{1}{z}\right.$$

$$-2(\delta_1-\delta_2)\frac{a^2+b^2}{a^2-b^2}\left\{\sqrt{R^4-c^2z^2}-R^2+\frac{c^2z^2}{4R^2}\right\}\frac{1}{z}$$

$$\left.-(\delta_1-\delta_2)(a^2+b^2)\left\{\frac{R^2}{\sqrt{R^4-c^2z^2}}-\frac{1}{2}\quad\frac{z}{R^2}\right\}\right],$$

$$\psi_1(z) = \frac{\mu}{k+1}\left\{(\delta_1-\delta_2)z+2(\delta_1+\delta_2)\frac{a-b}{a+b}z+(\delta_1-\delta_2)\frac{(a-b)^2}{(a+b)^2}z\right\}$$

$$+\frac{\mu}{k+1}\frac{2ab}{a^2-b^2}\left[(\delta_1-\delta_2)\left\{\sqrt{R^4-c^2z^2}-\frac{R^8}{(R^4-c^2z^2)^{3/2}}\right\}\frac{1}{z}\right.$$

$$-2(\delta_1+\delta_2)\left\{\frac{R^6}{(R^4-c^2z^2)^{1/2}}-R^4-\frac{c^2z^2}{2}\right\}\frac{1}{z^3}$$

$$-2(\delta_1-\delta_2)\frac{a^2+b^2}{a^2-b^2}\left\{\frac{R^6}{(R^4-c^2z^2)^{1/2}}-R^4-\frac{c^2z^2}{2}\right\}\frac{1}{z^3}$$

$$\left.+(\delta_1-\delta_2)(a^2+b^2)\left\{\frac{R^6}{(R^4-c^2z^2)^{3/2}}-1\right\}\frac{1}{z}\right].$$

For the shell

$$\phi_s(z) = \frac{\mu}{k+1}\frac{2ab}{a^2-b^2}\left[(\delta_1-\delta_2)\left\{(z-\sqrt{z^2-c^2})+\left(\frac{R^2}{(R^4-c^2z^2)^{1/2}}-1\right)z\right\}\right.$$

$$-2(\delta_1+\delta_2)\left\{\sqrt{R^4-c^2z^2}-R^2+\frac{c^2z^2}{4R^2}\right\}\frac{1}{z}-2(\delta_1-\delta_2)\frac{a^2+b^2}{a^2-b^2}\quad x$$

$$\left(\sqrt{R^4 - c^2z^2} - R^2 + \frac{c^2z^2}{4R^2}\right)\frac{1}{z} - (\delta_1-\delta_2)(a^2+b^2)\left(\frac{R^2}{\sqrt{R^4-c^2z^2}} - \frac{1}{2}\right)\frac{z}{R^2}\Bigg]$$

$$\psi_s(z) = -\frac{\mu}{k+1}\frac{2ab}{a^2-b^2}\Bigg[2(\delta_1+\delta_2)(z-\sqrt{z^2-c^2}) + (\delta_1-\delta_2)\frac{a^2+b^2}{a^2-b^2} \times$$

$$\left\{2z - \frac{z^2}{\sqrt{z^2-c^2}} - \sqrt{z^2-c^2}\right\}\Bigg] + \frac{\mu}{k+1}\frac{2ab}{a^2-b^2}\Bigg[(\delta_1-\delta_2) \times$$

$$\left\{\sqrt{R^4-c^2z^2} - \frac{R^8}{(R^4-c^2z^2)^{3/2}}\right\}\frac{1}{z} - 2(\delta_1+\delta_2)\left\{\frac{R^6}{\sqrt{R^4-c^2z^2}} - R^4\right.$$

$$\left.-\frac{c^2z^2}{2}\right\}\frac{1}{z^3} - 2(\delta_1-\delta_2)\frac{a^2+b^2}{a^2-b^2}\left\{\frac{R^6}{\sqrt{R^4-c^2z^2}} - R^4 - \frac{c^2z^2}{2}\right\}\frac{1}{z^3}$$

$$+ (\delta_1-\delta_2)(a^2+b^2)\left\{\frac{R^6}{(R^4-c^2z^2)^{3/2}} - 1\right\}\frac{1}{z}\Bigg]. \tag{6.15}$$

These results check with those obtained by S.C. Gupta in [14].

The particular case of a rotating circular disc with eccentric circular insert may be derived by taking $\delta_1 = \delta_2 = \delta$ and taking the limit $b \to a$. In the case of the insert, we get

$$\phi_1(z) = \frac{3+\nu}{16}\rho\,\omega^2R^2z - \frac{2\mu\delta}{k+1}(z-d) + \frac{2\mu\delta}{k+1}\frac{R^2+dz}{R^2-dz}\frac{a^2z}{R^2},$$

$$\psi_1(z) = \frac{2\mu\delta d}{k+1} - \frac{4\mu\delta}{k+1}\frac{a^2d(2R^2-dz)}{(R^2-dz)^2} \tag{6.16}$$

and for the shell

$$\phi_s(z) = \frac{3+\nu}{16} \varrho\, \omega^2 R^2 z + \frac{2\mu\delta}{k+1} \frac{R^2+dz}{R^2-dz} \frac{a^2 z}{R^2} ,$$

$$\psi_s(z) = -\frac{4\mu\delta}{k+1} \frac{a^2}{z-d} - \frac{4\mu\delta}{k+1} \frac{a^2 d(2R^2-dz)}{(R^2-dz)^2} . \tag{6.17}$$

If we put $\omega = 0$ in the above equations (6.16) and (6.17), we get the case of an eccentric circular inclusion in a circular medium. These results agree with those given in [13].

The stress functions for the problem of a composite rotating disc may be derived from the equations (6.16) and (6.17) by equating d to zero. The stresses calculated in this case agree with those obtained in (2.18).

CHAPTER 7

Composite Rotating Sphere

In this chapter, the problem of a rotating sphere containing a concentric inhomogenity is considered. The body force due to the mass of the body is neglected. The only body force supposed to be acting on the sphere is the inertia force. The elastic constants of the inhomogenity are different from those of the shell. The outer boundary of the shell is free from tractions. The problem is solved with the help of spherical solid harmonics.

Let the outer radius of the shell be b and let it contain a solid concentric spherical inhomogenity of radius a ($< b$). Thus one might imagine that there is a shell of outer radius b and inner radius a, in which an inhomogenity of radius a is inserted. The outer surface of the inhomogenity is welded to the inner surface of the shell to avoid slipping. The density of the inhomogenity is ρ' and that of the shell is ρ and the Lam'e constants for these materials are λ', μ' and λ , μ respectively. The common centre O is taken as the origin and any three mutually perpendicular lines are taken as the axes. The composite sphere is rotating about the Z-axis, with a constant angular velocity, say ω . Because the elastic constants of the composite sphere are different and since

there is an inertia force, stresses develop both in the inhomogenity and in the shell.

The body force is the inertia force and its components in X, Y, Z directions are given by $F_x = \rho\,\omega^2 x$, $F_y = \rho\,\omega^2 y$ and $F_z = 0$. This body force can be expressed as the gradient of a potential $\phi = \frac{1}{2}\,\omega^2(x^2+y^2)$ which may be written as

$$\phi = \frac{1}{3}\,\omega^2 r^2 + \frac{1}{6}\,\omega^2\,(x^2 + y^2 - 2z^2) \qquad (7.1)$$

where $r^2 = x^2 + y^2 + z^2$. For later use in the chapter, we put

$$\phi_1 = \frac{1}{3}\,\omega^2 r^2 \quad \text{and} \quad \phi_2 = \frac{\omega^2}{6}\,(x^2 + y^2 - 2z^2) \qquad (7.2)$$

whence

$$\phi = \phi_1 + \phi_2 \,.$$

The first term on the right hand side of (7.1) gives a purely radial force directed towards the centre of the sphere. The second term in the expression for ϕ in (7.1) is a homogeneous function of x, y, z of second degree and it satisfies the Laplaces equation. This is a spherical solid harmonic of second degree and it would be denoted by ϕ_2. At this stage, the problem may be divided into two separate problems : one with the body force which is the gradient of the potential given by the first term in (7.1) or by ϕ_1 in (7.2) and the second with the body force which is the gradient of the potential given by the second term in (7.1) or by ϕ_2 in (7.2). The solution is obtained by the principle of super-

position.

For the solution of both the problems, the equilibrium equations, when written in terms of displacements, are more useful. For the first problem, the spherical polar coordinates are used and the equilibrium equations are also written in that coordinate system. For the second problem, the Cartesian coordinates are used and it is solved using the spherical solid harmonics. As stated earlier, the solution of the present problem is obtained by superposing the solutions.

As remarked in the preceding paragraphs, it is expedient to write the equilibrium equations in terms of displacements only. To this end, we refer to the first chapter, where it is stated that in theory of elasticity, the unknown quantities to be determined, in any problem, are the six components P_{ij} $(i,j = x,y,z)$, the strain components e_{ij} $(i,j = x,y,z)$ and the three displacement components u,v,w. These fifteen unknowns are to be determined from the fifteen basic equations (1.1, 1.4, 1.5) subject to the boundary conditions (1.3). The equilibrium equations (1.1) are expressed in terms of stresses. However, in certain problems, as in the present one, it is more convenient to deal with equations in terms of displacement components. These are usually called the Navier Stokes equations. This is done as follows. We substitute the values of P_{ij} $(i,j = x,y,z)$ from equations (1.5) into equations (1.1), and then put the values of e_{ij} $(i,j = x,y,z)$ in terms of displacements from equations (1.4). After some simplification, the

following equations are obtained.

$$(\lambda + \mu)\frac{\partial\theta}{\partial x} + \mu\ \nabla^2 u + \rho F_x = 0\ ,$$

$$(\lambda + \mu)\frac{\partial\theta}{\partial y} + \mu\ \nabla^2 v + \rho F_y = 0\ ,$$

$$(\lambda + \mu)\frac{\partial\theta}{\partial z} + \mu\ \nabla^2 w + \rho F_z = 0 \tag{7.3}$$

where (F_x, F_y, F_z) are the body force components and $\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$.
In terms of vector notation, the above equations may be written as

$$(\lambda + \mu)\ \mathrm{grad}\ \theta + \mu\nabla^2(u,v,w) + \rho(F_x, F_y, F_z) = 0 \tag{7.4}$$

which is more useful, when the equations are to be written in other coordinate system. It may be noted that ∇^2 stands for the Laplacian operator.

It is clear that the elasticity problem is completely solved if one obtains the solution of (7.3) or equivalently (7.4), subject to the appropriate boundary conditions. Note that we need not adjoin the compatability equations, for the only purpose of the latter is to impose restrictions on the strain components that shall ensure single-valued continuous displacements u, v, w.

In spherical polar coordinates the equation (7.4) may be written as the following three equations.

$$(\lambda+2\mu) r \sin\theta \frac{\partial\Delta}{\partial r} - 2\mu\left\{\frac{\partial}{\partial\theta}(\ \omega_\phi \sin\theta) - \frac{\partial}{\partial\phi}(\omega_\theta\ )\right\} + \rho F_r = 0\ ,$$

$$(\lambda+2\mu)\ \sin\theta\,\frac{\partial\Delta}{\partial\theta} - 2\mu\left\{\frac{\partial\omega_r}{\partial\phi} - \frac{\partial}{\partial r}(r\,\omega_\phi\sin\theta)\right\} + \rho F_\theta = 0\ ,$$

$$(\lambda+2\mu)\ \frac{1}{\sin\theta}\ \frac{\partial\Delta}{\partial\phi} - 2\mu\left\{\frac{\partial}{\partial r}(r\,\omega_\theta) - \frac{\partial\omega_r}{\partial\theta}\right\} + \rho F_\phi = 0$$

(7.5)

where ω_r, ω_θ, ω_ϕ are the components of rotation and Δ is the cubical dilatation given by

$$\Delta = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2u_r\sin\theta) + \frac{\partial}{\partial\theta}(ru_\theta\sin\theta) + \frac{\partial}{\partial\phi}(ru_\phi)\right\}.$$

The rotational components ω_r, ω_θ, ω_ϕ are given by

$$\omega_r = \frac{1}{2r^2\sin\theta}\left\{\frac{\partial}{\partial\theta}(ru_\phi\sin\theta) - \frac{\partial}{\partial\phi}(ru_\theta)\right\}\ ,$$

$$\omega_\theta = \frac{1}{2r\sin\theta}\left\{\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}(ru_\phi\sin\theta)\right\},$$

$$\omega_\phi = \frac{1}{2r}\left\{\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial}{\partial\theta}(u_r)\right\}.$$

(7.6)

Here u_r, u_θ, u_ϕ are the displacements in the directions of r,θ,ϕ.

The above results (7.5) and (7.6) are sufficient to arrive at the solution of the problem. As remarked earlier in this chapter, we shall discuss two cases separately : one in which the potential is given by $\phi_1 = \frac{1}{3}\,\omega^2r^2$ and another in which the potential is given by $\phi_2 = \frac{\omega^2}{6}(x^2 + y^2 - 2z^2)$.

Case (1)

When the potential $\phi_1 = \frac{1}{3}\,\omega^2r^2$, the body force components may be written in spherical polar coordinates as

$$\frac{\partial}{\partial r}\left(\frac{1}{3}\omega^2 r^2\right), \quad \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{3}\omega^2 r^2\right), \quad \frac{1}{r\sin\theta}\ \frac{\partial}{\partial\phi}\left(\frac{1}{3}\omega^2 r^2\right)$$

or ($\frac{2}{3}\omega^2 r$, 0, 0) which shows that the body force is purely radial. By symmetry the displacement u_r shall be function of r only and u_θ, u_ϕ shall be identically zero. The last two equilibrium equations in (7.5) are identically satisfied. The first equation $(7.5)_1$ gives

$$\frac{d^2u_r}{dr^2} + \frac{2}{r}\frac{du_r}{dr} - \frac{2u_r}{r^2} + \frac{2}{3}\ \frac{\rho\,\omega^2}{(\lambda+2\mu)}\ r = 0 \tag{7.7}$$

where the partial derivatives with respect to r is replaced by total derivatives as u_r is a function of r only. The solution of the equation (7.7) is

$$u_r = Fr + \frac{E}{r^2} - \frac{\rho\,\omega^2}{15}\ \frac{r^3}{(\lambda + 2\mu)} \tag{7.8}$$

where F and E are constants. In the case of the shell, both the constants in (7.8) are admissible. To distinguish the constants for the shell from those of the inhomogenity, we replace F and E by D_1 and D_2.

$$u_r = D_1 r + \frac{D_2}{r^2} - \frac{\rho\,\omega^2}{15}\ \frac{r^3}{(\lambda + 2\mu)}\ . \tag{7.9}$$

Using strain displacement relations, one can find the strains and by Hooke's law, the stresses. The corresponding stress components P_{rr}, $P_{\theta\theta}$, $P_{\phi\phi}$ are

$$P_{rr} = D_1(3\lambda + 2\mu) - \frac{4\mu D_2}{r^3} - \frac{\rho\omega^2}{15} \frac{r^2(5\lambda + 6\mu)}{\lambda + 2\mu} ,$$

$$P_{\theta\theta} = D_1(3\lambda + 2\mu) + \frac{2\mu D_2}{r^3} - \frac{\rho\omega^2}{15} \frac{r^2(5\lambda + 2\mu)}{\lambda + 2\mu} ,$$

$$P_{\phi\phi} = P_{\theta\theta} , \quad P_{r\theta} = 0 , \quad P_{\theta\phi} = 0 , \quad P_{r\phi} = 0 . \tag{7.10}$$

In the case of the inhomogenity, the constant E becomes zero because, otherwise the displacement will tend to infinity as $r \rightarrow 0$. Hence the solution for the inhomogenity may be taken as

$$u_r = D_3 r - \frac{\rho'\omega^2}{15} \frac{r^3}{(\lambda' + 2\mu')} \tag{7.11}$$

where we have replaced F by another constant, say D_3. The corresponding stresses are

$$P_{rr} = D_3(3\lambda' + 2\mu') - \frac{\rho'\omega^2}{15} \frac{r^2(5\lambda' + 6\mu')}{\lambda' + 2\mu'} ,$$

$$P_{\theta\theta} = D_3(3\lambda' + 2\mu') - \frac{\rho'\omega^2}{15} \frac{r^2(5\lambda' + 2\mu')}{\lambda' + 2\mu'} ,$$

$$P_{\phi\phi} = P_{\theta\theta}, \quad P_{r\theta} = 0 , \quad P_{r\phi} = 0 , \quad P_{\theta\phi} = 0 . \tag{7.12}$$

To evaluate the constants in (7.9) and (7.11), we apply the boundary conditions. Note that because of symmetry for Φ_1, we need consider only the continuity of P_{rr} and u_r for this particular case.

1) Since the outer boundary of the shell is free from the tractions, therefore $P_{rr} = 0$ at $r = b$.

2) Since the inhomogenity and the shell form a continuous medium, P_{rr} is continuous at $r = a$.

3) The reason given in 2) applies for the displacement also. Therefore, u_r is continuous at $r = a$.

We use the equation $(7.10)_1$ for the condition 1) and $(7.10)_1$, $(7.12)_1$ for condition 2) and (7.9), (7.11) for condition 3). We get three equations in three unknowns D_1, D_2 and D_3. We solve them and get

$$D_1 = \frac{4\mu a^3 \left[\frac{2}{15} \varrho' \omega^2 a^2 + \frac{\varrho \omega^2}{15} \frac{1}{\lambda + 2\mu} \left\{ \frac{1}{4\mu a^3} b^5 (5\lambda + 6\mu)(3\lambda' + 2\mu' + 4\mu) + a^2 (3\lambda' + 2\mu' - 5\lambda - 6\mu) \right\} \right]}{4\mu a^3 \left\{ 3(\lambda' - \lambda) + 2(\mu' - \mu) \right\} + b^3 (3\lambda + 2\mu)(3\lambda' + 2\mu' + 4\mu)} ,$$

$$D_2 = \frac{b^3}{4\mu} \left\{ D_1 (3\lambda + 2\mu) - \frac{\varrho \omega^2}{15} \frac{b^2 (5\lambda + 6\mu)}{\lambda + 2\mu} \right\} ,$$

$$D_3 = \frac{\left[\frac{\varrho' \omega^2 a^2}{15(\lambda' + 2\mu')} \left\{ 4\mu a^5 (5\lambda' + 6\mu' - 3\lambda - 2\mu) + a^2 b^3 (5\lambda' + 6\mu' + 4\mu)(3\lambda + 2\mu) \right\} - \frac{\varrho \omega^2}{15} \left\{ 8\mu a^5 + 5a^2 b^3 (3\lambda + 2\mu) - 3b^5 (5\lambda + 6\mu) \right\} \right]}{4\mu a^3 \left\{ 3(\lambda' - \lambda) + 2(\mu' - \mu) \right\} + b^3 (3\lambda + 2\mu)(3\lambda' + 2\mu' + 4\mu)} .$$

(7.13)

The stresses and displacements may now be obtained by substituting the above values of D_1, D_2 and D_3 in (7.10), (7.9), (7.12) and (7.11)

which give the solution for the case i .

Case (ii)

When the potential $\phi_2 = \frac{\omega^2}{6}(x^2 + y^2 - 2z^2)$, the body force components are given by

$$\frac{\partial}{\partial x}\left\{\frac{\omega^2}{6}(x^2+y^2-2z^2)\right\},\ \frac{\partial}{\partial y}\left\{\frac{\omega^2}{6}(x^2+y^2-2z^2)\right\},\ \frac{\partial}{\partial z}\left\{\frac{\omega^2}{6}(x^2+y^2-2z^2)\right\}.$$

In this case, it is useful to use the Navier Stoke's equations in Cartesian coordinates :

$$(\lambda+\mu)\frac{\partial\theta}{\partial x} + \mu\nabla^2 u + \rho\frac{\partial}{\partial x}\left\{\frac{\omega^2}{6}(x^2+y^2-2z^2)\right\} = 0,$$

$$(\lambda+\mu)\frac{\partial\theta}{\partial y} + \mu\nabla^2 v + \rho\frac{\partial}{\partial y}\left\{\frac{\omega^2}{6}(x^2+y^2-2z^2)\right\} = 0,$$

$$(\lambda+\mu)\frac{\partial\theta}{\partial z} + \mu\nabla^2 w + \rho\frac{\partial}{\partial z}\left\{\frac{\omega^2}{6}(x^2+y^2-2z^2)\right\} = 0.$$

(7.14)

As is apparent, the last term makes the equations non-homogeneous. At this stage, it is useful to use ϕ_2 for $\frac{\omega^2}{6}(x^2+y^2-2z^2)$ in the last terms of the three equations in (7.14). Then the equations (7.14) may be written as

$$(\lambda+\mu)\frac{\partial\theta}{\partial x} + \mu\nabla^2 u + \rho\frac{\partial\phi_2}{\partial x} = 0,$$

$$(\lambda+\mu)\frac{\partial\theta}{\partial y} + \mu\nabla^2 v + \rho\frac{\partial\phi_2}{\partial y} = 0,$$

$$(\lambda+\mu)\frac{\partial\theta}{\partial z} + \mu\nabla^2 w + \rho\frac{\partial\phi_2}{\partial z} = 0.$$

(7.15)

A particular solution of (7.15) can be obtained by assuming

$$u^{(1)} = \frac{\partial \Delta}{\partial x} , \quad v^{(1)} = \frac{\partial \Delta}{\partial y} , \quad w^{(1)} = \frac{\partial \Delta}{\partial z} \tag{7.16}$$

where Δ is yet to be determined. Substituting (7.16) in (7.15) it may be seen that

$$(\lambda + 2\mu) \nabla^2 \Delta + \rho \phi_2 = 0$$

whose solution is, by a known property of spherical harmonics [9],

$$\Delta = - \frac{\rho}{14(\lambda + 2\mu)} r^2 \phi_2 .$$

Then the displacements are given by

$$u^{(1)} = - \frac{\rho}{14(\lambda + 2\mu)} \frac{\partial}{\partial x} (r^2 \phi_2) ,$$

$$v^{(1)} = - \frac{\rho}{14(\lambda + 2\mu)} \frac{\partial}{\partial y} (r^2 \phi_2) ,$$

$$w^{(1)} = - \frac{\rho}{14(\lambda + 2\mu)} \frac{\partial}{\partial z} (r^2 \phi_2) . \tag{7.17}$$

The tractions come out to be

$$p_{rx}^{(1)} = - \frac{\mu}{r} \frac{\rho}{(\lambda+2\mu)} \left(\frac{3r^2}{7} \frac{\partial \phi_2}{\partial x} + \frac{(7\lambda + 6\mu)}{7\mu} x\phi_2 \right) ,$$

$$p_{ry}^{(1)} = - \frac{\mu}{r} \frac{\rho}{(\lambda+2\mu)} \left(\frac{3r^2}{7} \frac{\partial \phi_2}{\partial y} + \frac{(7\lambda + 6\mu)}{7\mu} y\phi_2 \right) ,$$

$$p_{rz}^{(1)} = - \frac{\mu}{r} \frac{\rho}{(\lambda+2\mu)} \left(\frac{3r^2}{7} \frac{\partial \phi_2}{\partial z} + \frac{(7\lambda + 6\mu)}{7\mu} z\phi_2 \right) . \tag{7.18}$$

Note that the displacements (7.17) and the tractions (7.18) obtained are the particular solutions in the case of the shell. The corresponding displacements and tractions for the inhomogenity may be obtained by simply changing λ, μ, ϱ to λ', μ', ϱ' respectively in (7.17) and (7.18). We shall obtain the solution for the homogeneous equations separately, for the shell and for the inhomogenity. We begin the case for the shell.

Let $u^{(2)}$, $v^{(2)}$, $w^{(2)}$ be a solution of the homogeneous problem when the last terms on the left hand side of (7.15) are zero, i.e., $u^{(2)}$, $v^{(2)}$, $w^{(2)}$ satisfy the equations

$$(\lambda + \mu)\,\frac{\partial\theta^{(2)}}{\partial x} + \mu\,\nabla^2 u^{(2)} = 0\,,$$

$$(\lambda + \mu)\,\frac{\partial\theta^{(2)}}{\partial y} + \mu\,\nabla^2 v^{(2)} = 0\,,$$

$$(\lambda + \mu)\,\frac{\partial\theta^{(2)}}{\partial z} + \mu\,\nabla^2 w^{(2)} = 0 \tag{7.19}$$

where $\theta^{(2)} = \frac{\partial u^{(2)}}{\partial x} + \frac{\partial v^{(2)}}{\partial y} + \frac{\partial w^{(2)}}{\partial z}$. We obtain two sets of solutions of the above equations and their sum also will be the solution because of a well known theorem that if u_1 and u_2 are linearly independent solutions of a system of homogeneous equations, so also is their sum [15]. Consider a solution of (7.19) defined by the equations

$$(u^{(2)})_1 = r^2\,\frac{\partial W}{\partial x} + \alpha x W\,,$$

$$(v^{(2)})_1 = r^2 \frac{\partial W}{\partial y} + \alpha y W ,$$

$$(w^{(2)})_1 = r^2 \frac{\partial W}{\partial z} + \alpha z W$$

(7.20)

where α is a unknown constant and W is a spherical solid harmonic. For the case of a shell, let us choose ,

$$W = (A_1 + \frac{A_2}{r^5})\, \phi_2 \tag{7.21}$$

where A_1 and A_2 are some constants. It may be seen that the displacement components given by (7.20) satisfy (7.19) provided

$$\alpha = - \frac{2(2\lambda + 7\mu)}{(5\lambda + 7\mu)} . \tag{7.22}$$

Substituting the values of W and α from (7.21) and (7.22), in (7.20), we get

$$(u_s^{(2)})_1 = (A_1 r^2 + \frac{A_2}{r^3})\, \frac{\partial \phi_2}{\partial x} - \frac{5A_2 x \phi_2}{r^5} - 2(A_1 + \frac{A_2}{r^5})\frac{(2\lambda + 7\mu)}{(5\lambda + 7\mu)}\, x\phi_2 ,$$

$$(v_s^{(2)})_1 = (A_1 r^2 + \frac{A_2}{r^3})\, \frac{\partial \phi_2}{\partial y} - \frac{5A_2 y \phi_2}{r^5} - 2(A_1 + \frac{A_2}{r^5})\frac{(2\lambda + 7\mu)}{(5\lambda + 7\mu)}\, y\phi_2 ,$$

$$(w_s^{(2)})_1 = (A_1 r^2 + \frac{A_2}{r^3})\, \frac{\partial \phi_2}{\partial z} - \frac{5A_2 z \phi_2}{r^5} - 2(A_1 + \frac{A_2}{r^5})\frac{(2\lambda + 7\mu)}{(5\lambda + 7\mu)}\, z\phi_2$$

(7.23)

where the subscript s refers to the shell.

Consider another set of displacement components expressed by the equations

$$(u_s^{(2)})_2 = \frac{\partial U}{\partial x} \quad , \quad (v_s^{(2)})_2 = \frac{\partial U}{\partial y} , \quad (w_s^{(2)})_2 = \frac{\partial U}{\partial z} \tag{7.24}$$

where U is a spherical solid harmonic. We choose, in the case of shell

$$U = (B_1 + \frac{B_2}{r^5}) \phi_2 \tag{7.25}$$

where B_1 and B_2 are constants. The displacements (7.24) satisfy the equations (7.19) identically. Thus putting the value of U from (7.25) in (7.24), the displacement components $(u_s^{(2)})_2$, $(v_s^{(2)})_2$ and $(w_s^{(2)})_2$ are :

$$(u_s^{(2)})_2 = (B_1 + \frac{B_2}{r^5}) \frac{\partial \phi_2}{\partial x} - \frac{5B_2}{r^7} x\phi_2 ,$$

$$(v_s^{(2)})_2 = (B_1 + \frac{B_2}{r^5}) \frac{\partial \phi_2}{\partial y} - \frac{5B_2}{r^7} y\phi_2 ,$$

$$(w_s^{(2)})_2 = (B_1 + \frac{B_2}{r^5}) \frac{\partial \phi_2}{\partial z} - \frac{5B_2}{r^7} z\phi_2 . \tag{7.26}$$

We take a solution of the homogeneous equations (7.19) as the sum of (7.23) and (7.26). Hence a solution of the non-homogeneous equations (7.15) in the case of shell may be taken to be the sum of (7.17), (7.23) and (7.26). These come out to be

$$u_s = \Big[\Big\{A_1 r^2 + \frac{A_2}{r^3} + B_1 + \frac{B_2}{r^5} - \frac{\varrho}{14(\lambda+2\mu)} r^2\Big\} \frac{\partial \phi_2}{\partial x}$$

$$+ \Big\{- \frac{5A_2}{r^5} - \frac{2(2\lambda+7\mu)}{(5\lambda+7\mu)} (A_1 + \frac{A_2}{r^5}) - \frac{5B_2}{r^7} - \frac{\varrho}{7(\lambda+2\mu)}\Big\} x\phi_2\Big] ,$$

$$v_s = \left[\left\{A_1 r^2 + \frac{A_2}{r^3} + B_1 + \frac{B_2}{r^5} - \frac{\varrho}{14(\lambda+2\mu)} r^2\right\} \frac{\partial \phi_2}{\partial y}\right.$$

$$\left. + \left\{- \frac{5A_2}{r^5} - \frac{2(2\lambda+7\mu)}{(5\lambda+7\mu)} \left(A_1 + \frac{A_2}{r^5}\right) - \frac{5B_2}{r^7} - \frac{\varrho}{7(\lambda+2\mu)}\right\} \; y\phi_2\right],$$

$$w_s = \left[\left\{A_1 r^2 + \frac{A_2}{r^3} + B_1 + \frac{B_2}{r^5} - \frac{\varrho}{14(\lambda+2\mu)} r^2\right\} \frac{\partial \phi_2}{\partial z}\right.$$

$$\left. + \left\{- \frac{5A_2}{r^5} - \frac{2(2\lambda+7\mu)}{(5\lambda+7\mu)} \left(A_1 + \frac{A_2}{r^5}\right) - \frac{5B_2}{r^7} - \frac{\varrho}{7(\lambda+2\mu)}\right\} \; z\phi_2\right].$$

(7.27)

Across a surface r = constant in the shell, the tractions come out to be

$$p^s_{rx} = \frac{1}{r} \left[\left\{\left(A_1 r^2 + \frac{A_2}{r^3}\right) \frac{2\mu(8\lambda+7\mu)}{(5\lambda+7\mu)} + 2\mu B_1 + \frac{2\mu B_2}{r^5} - \frac{3\mu\varrho r^2}{7(\lambda+2\mu)}\right\} \frac{\partial \phi_2}{\partial x}\right.$$

$$\left. + \left\{- \frac{2\mu A_2}{r^5} \frac{59\lambda+49\mu}{5\lambda+7\mu} - \frac{2\mu(19\lambda+14\mu)}{(5\lambda+7\mu)} A_1 - \frac{10\mu B_2}{r^7} - \frac{(7\lambda+6\mu)\varrho}{7(\lambda+2\mu)}\right\} x\phi_2\right],$$

$$p^s_{ry} = \frac{1}{r} \left[\left\{\left(A_1 r^2 + \frac{A_2}{r^3}\right) \frac{2\mu(8\lambda+7\mu)}{(5\lambda+7\mu)} + 2\mu B_1 + \frac{2\mu B_2}{r^5} - \frac{3\mu\varrho r^2}{7(\lambda+2\mu)}\right\} \frac{\partial \phi_2}{\partial y}\right.$$

$$\left. + \left\{- \frac{2\mu A_2}{r^5} \frac{59\lambda+49\mu}{5\lambda+7\mu} - \frac{2\mu(19\lambda+14\mu)}{(5\lambda+7\mu)} A_1 - \frac{10\mu B_2}{r^7} - \frac{(7\lambda+6\mu)\varrho}{7(\lambda+2\mu)}\right\} y\phi_2\right],$$

$$p^s_{rz} = \frac{1}{r} \left[\left\{\left(A_1 r^2 + \frac{A_2}{r^3}\right) \frac{2\mu(8\lambda+7\mu)}{(5\lambda+7\mu)} + 2\mu B_1 + \frac{2\mu B_2}{r^5} - \frac{3\mu\varrho r^2}{7(\lambda+2\mu)}\right\} \frac{\partial \phi_2}{\partial z}\right.$$

$$\left. + \left\{- \frac{2\mu A_2}{r^5} \frac{59\lambda+49\mu}{5\lambda+7\mu} - \frac{2\mu(19\lambda+14\mu)}{(5\lambda+7\mu)} A_1 - \frac{10\mu B_2}{r^7} - \frac{(7\lambda+6\mu)\varrho}{7(\lambda+2\mu)}\right\} z\phi_2\right].$$

(7.28)

This gives the solution for the case of the shell, where the constants A_1, A_2, B_1, B_2 are yet to be determined. This will be done later.

We now proceed to obtain a solution of the homogeneous equations (7.19) for the case of the inhomogenity. Consider the displacements defined by (7.20). Choosing

$$W = c_1 \phi_2 \tag{7.29}$$

where c_1 is a constant. We find that the displacements satisfy the equilibrium equations (7.19) for the same value of α as given in (7.22). In this case, substituting the value of W from (7.29) into (7.20), we get

$$
\begin{aligned}
(u_i^{(2)})_1 &= c_1 r^2 \frac{\partial \phi_2}{\partial x} - 2\,\frac{(2\lambda' + 7\mu')}{(5\lambda' + 7\mu')}\; x c_1 \phi_2 \;, \\
(v_i^{(2)})_1 &= c_1 r^2 \frac{\partial \phi_2}{\partial y} - 2\,\frac{(2\lambda' + 7\mu')}{(5\lambda' + 7\mu')}\; y c_1 \phi_2 \;, \\
(w_i^{(2)})_1 &= c_1 r^2 \frac{\partial \phi_2}{\partial z} - 2\,\frac{(2\lambda' + 7\mu')}{(5\lambda' + 7\mu')}\; z c_1 \phi_2 \;.
\end{aligned}
\tag{7.30}
$$

As stated, the displacements defined by (7.24) satisfy the equilibrium equations identically. Next, we choose another solution $(u_i^{(2)})_2$, $(v_i^{(2)})_2$, $(w_i^{(2)})_2$ for (7.19) and put

$$(u_i^{(2)})_2 = \frac{\partial U}{\partial x}\,, \quad (v_i^{(2)})_2 = \frac{\partial U}{\partial y}\,, \quad (w_i^{(2)})_2 = \frac{\partial U}{\partial z}\,.$$

Let

$$U = c_2\phi_2 \tag{7.31}$$

where c_2 is a constant and ϕ_2 is given by $(7.2)_1$. The displacement components are given by

$$(u_1^{(2)})_2 = c_2\,\frac{\partial\phi_2}{\partial x}\ ,\quad (v_1^{(2)})_2 = c_2\,\frac{\partial\phi_2}{\partial y}\ ,\quad (w_1^{(2)})_2 = c_2\,\frac{\partial\phi_2}{\partial z}\ . \tag{7.32}$$

A solution of the non-homogeneous problem in the case of the inhomogenity is obtained as the sum of (7.17), (7.30) and (7.32). These solutions are

$$u_1 = \left(c_1r^2+c_2-\frac{\rho'}{14(\lambda'+2\mu')}\,r^2\right)\frac{\partial\phi_2}{\partial x} + \left(-\,\frac{2(2\lambda'+7\mu')}{(5\lambda'+7\mu')}\,c_1-\frac{\rho'}{7(\lambda'+2\mu')}\right)x\phi_2\ ,$$

$$v_1 = \left(c_1r^2+c_2-\frac{\rho'}{14(\lambda'+2\mu')}\,r^2\right)\frac{\partial\phi_2}{\partial y} + \left(-\,\frac{2(2\lambda'+7\mu')}{(5\lambda'+7\mu')}\,c_1-\frac{\rho'}{7(\lambda'+2\mu')}\right)y\phi_2\ ,$$

$$w_1 = \left(c_1r^2+c_2-\frac{\rho'}{14(\lambda'+2\mu')}\,r^2\right)\frac{\partial\phi_2}{\partial z} + \left(-\,\frac{2(2\lambda'+7\mu')}{(5\lambda'+7\mu')}\,c_1-\frac{\rho'}{7(\lambda'+2\mu')}\right)z\phi_2\ . \tag{7.33}$$

The corresponding tractions are

$$p^1_{rx} = \frac{1}{r}\Bigg[\left\{\frac{2\mu'(8\lambda'+7\mu')}{(5\lambda'+7\mu')}\,c_1r^2 + 2\mu'c_2 - \frac{3\mu'}{7}\,\frac{\rho'}{(\lambda'+2\mu')}\,r^2\right\}\frac{\partial\phi_2}{\partial x}$$

$$+\left\{-\,\frac{2\mu'(19\lambda'+14\mu')}{(5\lambda'+7\mu')}\,c_1 - \frac{\rho'}{(\lambda'+2\mu')}\,\frac{(7\lambda'+6\mu')}{7}\right\}x\phi_2\Bigg]\ ,$$

$$p^1_{ry} = \frac{1}{r}\left[\left\{\frac{2\mu'(8\lambda'+7\mu')}{(5\lambda'+7\mu')} c_1 r^2 + 2\mu' c_2 - \frac{3\mu'}{7}\ \frac{\rho'}{(\lambda'+2\mu')} r^2\right\} \frac{\partial \phi_2}{\partial y}\right.$$

$$\left. + \left\{- \frac{2\mu'(19\lambda'+14\mu')}{(5\lambda'+7\mu')} c_1 - \frac{\rho'(7\lambda'+6\mu')}{7(\lambda'+2\mu')}\right\} y\phi_2\right] ,$$

$$p^1_{rz} = \frac{1}{r}\left[\left\{\frac{2\mu'(8\lambda'+7\mu')}{(5\lambda'+7\mu')} c_1 r^2 + 2\mu' c_2 - \frac{3\mu'}{7}\ \frac{\rho'}{(\lambda'+2\mu')} r^2\right\} \frac{\partial \phi_2}{\partial z}\right.$$

$$\left. + \left\{- \frac{2\mu'(19\lambda'+14\mu')}{(5\lambda'+7\mu')} c_1 - \frac{\rho'(7\lambda'+6\mu')}{7(\lambda'+2\mu')}\right\} z\phi_2\right] .$$

(7.34)

These give the tractions for the inhomogenity. These involve the constants c_1, c_2, which will be determined in the next chapter.

With these results, we are in a position to obtain the solution, which is discussed in the next chapter.

CHAPTER 8

Composite Rotating Sphere (Continued)

As seen in the last chapter, we have been able to evaluate the tractions and displacements which involve certain constants. These will be evaluated in this chapter by using the appropriate boundary conditions. The boundary conditions may be stated as follows.

1) P_{rx} , P_{ry} , P_{rz} are each zero at $r = b$,

2) P_{rx} , P_{ry} , P_{rz} are separately continuous at $r = a$,

3) u, v, w are separately continuous at $r = a$.

It so happens that the condition $P_{rx} = 0$ at $r = b$ gives the same equations as are obtained when P_{ry} and P_{rz} each is equal to zero at $r = b$. Similarly, the continuity of P_{rx} gives the same equations as the continuity of P_{ry} and P_{rz} . Lastly the continuity of u at $r = a$ gives the same equations as the continuity of v and w. Apparently these conditions give 3 equations. But if we equate the coefficients of $\frac{\partial \phi_2}{\partial x}$ and $x\phi_2$ or $y\phi_2$ or $z\phi_2$ separately in each of the above three equations, the following six equations are obtained.

$$\frac{A_1 b^2(8\lambda+7\mu)}{5\lambda+7\mu} + \frac{8\lambda+7\mu}{5\lambda+7\mu}\,\frac{A_2}{b^3} + B_1 + \frac{B_2}{b^5} = \frac{\rho}{\lambda+2\mu}\,\frac{3b^2}{14}\,, \tag{8.1}$$

$$\frac{A_2(59\lambda+49\mu)}{b^5(5\lambda+7\mu)} + \frac{19\lambda+14\mu}{5\lambda+7\mu}\,A_1 + \frac{5B_2}{b^7} = -\frac{\rho}{\lambda+2\mu}\,\frac{7\lambda+6\mu}{14\mu}\,, \tag{8.2}$$

$$a^2A_1 + \frac{A_2}{a^3} + B_1 + \frac{B_2}{a^5} - \frac{\rho}{\lambda+2\mu}\,\frac{a^2}{14} = c_1a^2 + c_2 - \frac{\rho'}{\lambda'+2\mu'}\,\frac{a^2}{14}\,, \tag{8.3}$$

$$-\frac{5A_2}{a^5} - \frac{2(2\lambda+7\mu)}{(5\lambda+7\mu)}\,\frac{A_2}{a^5} - \frac{2(2\lambda+7\mu)}{(5\lambda+7\mu)}\,A_1 - \frac{5B_2}{a^7} - \frac{\rho}{7(\lambda+2\mu)}$$

$$= -\frac{2(2\lambda'+7\mu')}{(5\lambda'+7\mu')}\,c_1 - \frac{\rho'}{7(\lambda'+2\mu')}\,, \tag{8.4}$$

$$\frac{2\mu(8\lambda+7\mu)}{(5\lambda+7\mu)}\,A_1a^2 + \frac{2\mu(8\lambda+7\mu)}{(5\lambda+7\mu)}\,\frac{A_2}{a^3} + 2\mu B_1 + \frac{2\mu B_2}{a^5} - \frac{\rho}{(\lambda+2\mu)}\,\frac{3\mu a^2}{7}$$

$$= \frac{2\mu'(8\lambda'+7\mu')}{(5\lambda'+7\mu')}\,c_1a^2 + 2\mu'c_2 - \frac{\rho'}{(\lambda'+2\mu')}\,\frac{3\mu'a^2}{7}\,, \tag{8.5}$$

$$-\frac{2\mu(59\lambda+49\mu)}{(5\lambda+7\mu)}\,\frac{A_2}{a^5} - \frac{2\mu(19\lambda+14\mu)}{(5\lambda+7\mu)}\,A_1 - \frac{10\mu B_2}{a^7} - \frac{\rho}{(\lambda+2\mu)}\,\frac{(7\lambda+6\mu)}{7}$$

$$= -\frac{2\mu'(19\lambda'+14\mu')}{(5\lambda'+7\mu')}\,c_1 - \frac{\rho'}{(\lambda'+2\mu')}\,\frac{(7\lambda'+6\mu')}{7}\,. \tag{8.6}$$

These equations are linear simultaneous equations in A_1, A_2, B_1, B_2, c_1, c_2 and may be directly solved. The values of these constants are given below.

$$
A_1 = \frac{
\begin{aligned}
&\Big[\frac{\rho}{\lambda+2\mu}\,\frac{5\lambda+7\mu}{7\mu}\Big\{2\mu(\mu'-\mu)\Big\{\mu'(19\lambda'+14\mu')-(7\lambda+6\mu)(2\lambda'+7\mu')\Big\}\\
&(19\lambda+14\mu)a^{10}+5\mu\Big\{2\mu\mu'(\lambda'-7\mu')(59\lambda+49\mu)-\mu\mu'(19\lambda'+14\mu')\\
&(103\lambda+161\mu)-\mu'(7\lambda+6\mu)(107\lambda\lambda'+175\lambda'\mu+70\lambda\mu'+98\mu\mu')\\
&+3\mu'^2(19\lambda'+14\mu')(13\lambda+21\mu)+2\mu(2\lambda'+7\mu')(56\lambda^2+274\lambda\mu+189\mu^2)\Big\}a^7b^3\\
&+7\Big\{(\lambda+3\mu)(\mu'-\mu)\Big\{2\mu(2\lambda'+7\mu')(59\lambda+49\mu)-\mu'(19\lambda'+14\mu')(29\lambda+49\mu)\Big\}\\
&+\mu(59\lambda+49\mu)\Big\{\mu'(19\lambda'+14\mu')-2\mu(2\lambda'+7\mu')\Big\}(\lambda+3\mu-\mu')\Big\}a^5b^5\\
&+(7\lambda+6\mu)\Big\{\mu'(19\lambda'+14\mu')-2\mu(2\lambda'+7\mu')\Big\}\Big\{2\mu'(2\lambda+7\mu)-\mu(19\lambda+14\mu)\Big\}b^{10}\\
&+5(\mu'-\mu)(56\lambda^2+274\lambda\mu+189\mu^2)\Big\{\mu'(19\lambda'+14\mu')-2\mu(2\lambda'+7\mu')\Big\}\Big\}a^3b^7\\
&+\rho'(5\lambda+7\mu)\Big\{4(\lambda'+\mu')(\mu'-\mu)(19\lambda+14\mu)a^{10}-(59\lambda+49\mu)\\
&\Big\{\mu'(19\lambda'+14\mu')-2\mu(2\lambda'+7\mu')\Big\}a^5b^5-5\Big\{6\mu\mu'(25\lambda+21\mu)-\mu'(3\lambda'+2\mu')\\
&(29\lambda+49\mu)-4\mu'(\lambda'+\mu')(5\lambda+7\mu)+4\lambda'\mu(8\lambda+7\mu)\Big\}a^7b^3\Big\}\Big]
\end{aligned}
}{
\begin{aligned}
&2\Big[2a^{10}(\mu'-\mu)(19\lambda+14\mu)\Big\{\mu(19\lambda+14\mu)(2\lambda'+7\mu')-\mu'(19\lambda'+14\mu')\\
&(2\lambda+7\mu)\Big\}+25\Big\{2\mu^2(8\lambda+7\mu)^2(2\lambda'+7\mu')-\mu\mu'(8\lambda+7\mu)(5\lambda+7\mu)(23\lambda'+28\mu')\\
&+\mu'^2(19\lambda'+14\mu')(5\lambda+7\mu)^2+1\overset{9}{\underset{\times}{8}}\lambda\lambda'\mu\mu'(\lambda+\mu)\Big\}a^7b^3+21(\lambda+\mu)(\mu'-\mu)\\
&\Big\{4\mu(2\lambda'+7\mu')(59\lambda+49\mu)-2\mu'(19\lambda'+14\mu')(44\lambda+49\mu)\Big\}a^5b^5\\
&+25\Big\{\mu'(19\lambda'+14\mu')-2\mu(2\lambda'+7\mu')\Big\}(\mu'-\mu)(8\lambda+7\mu)^2a^3b^7\\
&+(19\lambda+14\mu)\Big\{\mu'(19\lambda'+14\mu')-2\mu(2\lambda'+7\mu')\Big\}\Big\{\mu(19\lambda+14\mu)\\
&-2\mu'(2\lambda+7\mu)\Big\}b^{10}\Big]
\end{aligned}
},
$$

$$
A_2 = \dfrac{\begin{array}{l}
\Big[\dfrac{\rho(5\lambda+7\mu)}{7\mu(\lambda+2\mu)}\ \Big\{70(\mu'-\mu)(\lambda+2\mu)(4\lambda+3\mu)\big\{\mu'(19\lambda'+14\mu') \\
-2\mu(2\lambda'+7\mu')\big\}\ b^9 + 7\big\{\mu'(19\lambda'+14\mu')-2\mu(2\lambda'+7\mu')\big\}\big\{10\mu(\lambda+2\mu) \\
(4\lambda+3\mu)-\mu'(\lambda+2\mu)(21\lambda+16\mu)\big\}\ a^2b^7 + 14(\mu'-\mu)(\lambda+2\mu)\big\{10\mu(2\lambda'+7\mu') \\
(4\lambda+3\mu)-2\mu'(19\lambda'+14\mu')(\lambda+6\mu)\big\}\ a^7b^2 + 10\mu\mu'\ \big\{\mu(\lambda'-7\mu')(19\lambda+14\mu) \\
-14\mu(19\lambda'+14\mu')(\lambda+2\mu)-(7\lambda+6\mu)(16\lambda\lambda'+35\lambda\mu'+35\lambda'\mu+49\mu\mu') \\
+7\mu'(19\lambda'+14\mu')(\lambda+2\mu)\big\}\ a^9 + 140\mu^2(2\lambda'+7\mu')(4\lambda+3\mu)(\lambda+2\mu)\,a^9\Big\}a^3b^3 \\
+\ \rho'(5\lambda+7\mu)\,a^5b^3\ \Big\{84(\lambda+\mu)(\lambda'+\mu')(\mu-\mu')\,a^5b^2-(19\lambda+14\mu) \\
\big\{\mu'(19\lambda'+14\mu')-2\mu(2\lambda'+7\mu')\big\}\Big\}b^7 - 10a^7\Big\{(16\lambda\lambda'+28\mu\mu')(\mu-\mu') \\
+7\lambda'\mu(2\mu-5\mu')+7\lambda\mu'(5\mu-2\mu')\Big\}\Big]
\end{array}}{\begin{array}{l}
-2\ \Big[2a^{10}(\mu'-\mu)(19\lambda+14\mu)\ \big\{\mu(19\lambda+14\mu)(2\lambda'+7\mu')-\mu'(19\lambda'+14\mu') \\
(2\lambda+7\mu)\big\}\ +25\ \big\{2\mu^2(8\lambda+7\mu)^2(2\lambda'+7\mu')-\mu\mu'(8\lambda+7\mu)(5\lambda+7\mu)(23\lambda'+28\mu') \\
+\mu'^2(19\lambda'+14\mu')(5\lambda+7\mu)^2+1\overset{9}{\underset{\lambda}{8}}\lambda'\lambda\mu\mu'(\lambda+\mu)\big\}\ a^7b^3\ +21(\lambda+\mu)(\mu'-\mu) \\
\big\{4\mu(2\lambda'+7\mu')(59\lambda+49\mu)-2\mu'(19\lambda'+14\mu')(44\lambda+49\mu)\big\}\ a^5b^5 \\
+25\ \big\{\mu'(19\lambda'+14\mu')-2\mu(2\lambda'+7\mu')\big\}\ (\mu'-\mu)(8\lambda+7\mu)^2\ a^3b^7+(19\lambda+14\mu) \\
\big\{\mu'(19\lambda'+14\mu')-2\mu(2\lambda'+7\mu')\big\}\big\{\mu(19\lambda+14\mu)-2\mu'(2\lambda+7\mu)\big\}\ b^{10}\Big]
\end{array}},
$$

$$
B_1 = \frac{\rho}{\lambda+2\mu}\ \frac{(\lambda+3\mu)}{10\mu}\,b^2 - \frac{21(\lambda+\mu)}{5(5\lambda+7\mu)}\,A_1b^2 + \frac{(19\lambda+14\mu)}{5(5\lambda+7\mu)}\ \frac{A_2}{b^3}\ ,
$$

$$
B_2 = -\frac{b^7}{5}\ \left\{\frac{\rho}{\lambda+2\mu}\ \frac{7\lambda+6\mu}{14\mu} + \frac{(19\lambda+14\mu)}{5\lambda+7\mu}\,A_1 + \frac{(59\lambda+49\mu)}{(5\lambda+7\mu)}\ \frac{A_2}{b^5}\right\},
$$

$$c_1 = \frac{5\lambda'+7\mu'}{4\lambda'+14\mu'}\left[-\frac{\rho'}{7(\lambda'+2\mu')}+A_1\ \frac{2a^7(2\lambda+7\mu)-b^7(19\lambda+14\mu)}{(5\lambda+7\mu)\,a^7}\right.$$

$$\left.+\ \frac{(29\lambda+49\mu)\,a^2\ -\ (59\lambda+49\mu)\,b^2}{(5\lambda+7\mu)\,a^7}\ A_2+\frac{\rho}{\lambda+2\mu}\ \frac{2\mu a^7-b^7(7\lambda+6\mu)}{14\mu a^7}\right],$$

$$c_2 = \left[A_1\ \frac{10a^7\left\{(5\lambda+7\mu)(2\lambda'+7\mu')-(5\lambda'+7\mu')(2\lambda+7\mu)\right\}-42(\lambda+\mu)(2\lambda'+7\mu')a^5b^2+21b^7(\lambda'+\mu')(19\lambda+14\mu)}{10(5\lambda+7\mu)(2\lambda'+7\mu')a^5}\right.$$

$$+\ A_2\ \frac{2(19\lambda+14\mu)(2\lambda'+7\mu')a^5+5a^2b^3\left\{2(5\lambda+7\mu)(2\lambda'+7\mu')-(29\lambda+49\mu)(5\lambda'+7\mu')\right\}+21(\lambda'+\mu')(59\lambda+49\mu)b^5}{10(2\lambda'+7\mu')(5\lambda+7\mu)a^5b^3}$$

$$+\ \frac{\rho}{\lambda+2\mu}\ \frac{2(\lambda+3\mu)(2\lambda'+7\mu')a^5b^2+3(7\lambda+6\mu)(\lambda'+\mu')b^7-10\mu a^7(\lambda'+2\mu')}{20\mu a^5(2\lambda'+7\mu')}$$

$$\left.+\ \frac{\rho' a^2}{2(2\lambda'+7\mu')}\right]. \qquad (8.7)$$

Substituting these values of the constants in the appropriate equations, the displacements and tractions can be obtained in the inhomogenity and the shell. This gives the solution for case (ii).

We now revert back to the solution of the original problem which is actually arrived at by adding the solutions obtained in case (i) and case (ii) described on pages 98-114 ,

chapter 7. The displacements in the inhomogenity are obtained by adding the results in (7.11),(7.33). Similarly the displacements in the shell are obtained by adding the results obtained in (7.9) and (7.27). Hence the stress field is known everywhere in the shell and inhomogenity.

It may be mentioned that for each of the solutions that we have obtained, we have satisfied the conditions of continuity at the interface and the condition of tractions at free surface at the boundary. Fortunately, the outer boundary was free from tractions. The solution satisfies the Navier Stoke's equations with the appropriate body force components. Thus the sum of the displacements obtained from (7.11),(7.33) and (7.9),(7.27) will satisfy the Navier Stoke's equations and also the boundary conditions as stated above. The uniqueness theorem in the theory of elasticity guarantees that the solution will be the solution of the problem. We have evaluated only the displacements, but if desired, the strains and then the stresses can be directly found.

It is of some interest to know the stress components P_{rr}, $P_{r\theta}$, $P_{r\phi}$, for they give the normal and shearing stress components at the surface r = constant. It is to be noted that P_{rr}, $P_{r\theta}$, $P_{r\phi}$ are given by

$$P_{rr} = P_{rx} \sin\theta \cos\phi + P_{ry} \sin\theta \sin\phi + P_{rz} \cos\theta ,$$

$$P_{r\theta} = P_{rx} \cos\theta \cos\phi + P_{ry} \cos\theta \sin\phi - P_{rz} \sin\theta ,$$

$$P_{r\phi} = -P_{rx}\sin\phi + P_{ry}\cos\phi .$$

This gives the normal and shear stress components. It may be verified that at the interface $r = a$, P_{rr}, $P_{r\theta}$, $P_{r\phi}$ are continuous.

It is of some interest that the values of the radial displacement at the equilibrium boundary and the outer boundary are given by

$$u_r\Big|_{r=a} = D_3 a - \frac{\rho'\omega^2}{15}\,\frac{a^3}{(\lambda'+2\mu')} + \frac{\omega^2 a}{3}(\sin^2\theta - 2\cos^2\theta)\left[\left(c_1 a^2 + c_2 - \frac{\rho' a^2}{14(\lambda'+2\mu')}\right) + \frac{a^2}{2}\left\{-\frac{2(2\lambda'+7\mu')}{(5\lambda'+7\mu')}c_1 - \frac{\rho'}{7(\lambda'+2\mu')}\right\}\right]. \qquad (8.8)$$

$$u_r\Big|_{r=b} = D_1 b + \frac{D_2}{b^2} - \frac{\rho\omega^2}{15}\,\frac{b^3}{(\lambda+2\mu)} + \frac{\omega^2 b}{3}(\sin^2\theta - 2\cos^2\theta)\left[A_1 b^2 + \frac{A_2}{b^3} + B_1 + \frac{B_2}{b^5} - \frac{\rho}{(\lambda+2\mu)}\,\frac{b^2}{14} + \frac{b^2}{2}\left\{-\frac{5A_2}{b^5} - \frac{2(2\lambda+7\mu)}{(5\lambda+7\mu)}\left(A_1 + \frac{A_2}{b^5}\right) - \frac{5B_2}{b^7} - \frac{\rho}{7(\lambda+2\mu)}\right\}\right], \qquad (8.9)$$

where the values of the constants A_1, A_2, B_1, B_2, c_1 and c_2 are given by (8.7). It may be remarked that the displacement $(u_r)_{r=b}$ is minimum at $\theta = 0$ and maximum at $\theta = \pi/2$, i.e., at the equator of the sphere.

Some particular cases may be directly derived. For example by putting $\lambda' = \lambda$, $\mu' = \mu$ and $\rho' = \rho$, we get the case of a solid homogeneous sphere rotating about an axis. The results check with the known result [9]. Similarly the case of a rotating shell may be derived by putting $\lambda' = 0$ and $\mu' = 0$.

Numerical work for the calculation of stresses in the inhomogenity and in the shell has been done on Computer CDC 3600. The Poisson's ratio for the inhomogenity and for the shell were each taken to be 1/3. The outer radius of the sphere was 1.5 and the radius of the hole was taken to be 1. The value of ρ $^2/E$ was 0.005 and that of ρ' $^2/E$ was 0.006. The stresses were calculated for different values of E'/E. Four cases have been discussed with E'/E = 1/2, 2/3, 3/2, 2 respectively. The stresses P_{rr}, $P_{r\theta}$, $P_{r\phi}$ are calculated at the equilibrium boundary and they are tabulated.

The table 8.1 is for $E'/E = 1/2$. The first column in table 8.1 gives the angular distance θ of the point from the origin, measured from the Z-axis in the clockwise direction. The second, third and fourth columns give the values of the stresses P_{rr}, $P_{r\theta}$, $P_{r\phi}$. All these stresses are the same for the inhomogenity and the shell as they should be. In the tables 8.2, 8.3, 8.4 the same results are repeated for E'/E = 2/3, 3/2, 2 respectively.

It may be seen that as the ratio of Young's modulus of the insert to that of the shell, increases from 1/2 to 2, the stresses $\frac{P_{rr}}{E}$, $\frac{P_{r\theta}}{E}$ decrease which is obvious intutively.

TABLE 8.1

ED = 0.5

θ	P_{rr}/E'	$P_{r\theta}/E'$	$P_{r\phi}/E'$
0	0.0482	0.0000	0.0000
15	0.0435	0.0135	0.0000
30	0.0305	0.0234	0.0000
45	0.0129	0.0270	0.0000
60	-0.0047	0.0234	0.0000
75	-0.0176	0.0135	0.0000
90	-0.0223	0.0000	0.0000
105	-0.0176	-0.0135	0.0000
120	-0.0047	-0.0234	0.0000
135	0.0129	-0.0270	0.0000
150	0.0305	-0.0234	0.0000
165	0.0435	-0.0135	0.0000
180	0.0482	0.0000	0.0000

TABLE 8.2

ED = 2/3

θ	P_{rr}/E'	$P_{r\theta}/E'$	$P_{r\phi}/E'$
0	0.0079	0.0000	0.0000
15	0.0072	0.0026	0.0000
30	0.0055	0.0045	0.0000
45	0.0030	0.0052	0.0000
60	0.0006	0.0045	0.0000
75	-0.0012	0.0026	0.0000
90	-0.0019	0.0000	0.0000
105	-0.0012	-0.0026	0.0000
120	0.0006	-0.0045	0.0000
135	0.0030	-0.0052	0.0000
150	0.0055	-0.0045	0.0000
165	0.0072	-0.0026	0.0000
180	0.0079	0.0000	0.0000

TABLE 8.3

RD = 1.5

θ	P_{rr}/E	$P_{r\theta}/E$	$P_{r\phi}/E$
0	-0.0030	0.0000	0.0000
15	-0.0025	0.0001	0.0000
30	-0.0012	0.0001	0.0000
45	0.0006	0.0001	0.0000
60	0.0024	0.0001	0.0000
75	0.0038	0.0001	0.0000
90	0.0043	0.0000	0.0000
105	0.0038	-0.0001	0.0000
120	0.0024	-0.0001	0.0000
135	0.0006	-0.0001	0.0000
150	-0.0012	-0.0001	0.0000
165	-0.0025	-0.0001	0.0000
180	-0.0030	0.0000	0.0000

TABLE 8.4

ED = 2.0

θ	P_{rr}/E	$P_{r\theta}/E$	$P_{r\phi}/E$
0	-0.0047	0.0000	0.0000
15	-0.0040	-0.0002	0.0000
30	-0.0022	-0.0004	0.0000
45	0.0003	-0.0005	0.0000
60	0.0028	-0.0004	0.0000
75	0.0046	-0.0002	0.0000
90	0.0053	0.0000	0.0000
105	0.0046	0.0002	0.0000
120	0.0028	0.0004	0.0000
135	0.0003	0.0005	0.0000
150	-0.0022	0.0004	0.0000
165	-0.0040	0.0002	0.0000
180	-0.0047	0.0000	0.0000

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